

COMBINATORICS OF COMBINATORIAL TOPOLOGY

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ABSTRACT. We develop a tighter implementation of basic PL topology, which keeps track of some combinatorial structure beyond PL homeomorphism type. With this technique we clarify some aspects of PL transversality and give combinatorial proofs of a number of known results.

New results include a combinatorial characterization of collapsible polyhedra in terms of constructible posets (generalizing face posets of constructible simplicial complexes in the sense of Hochster). The relevant constructible posets are also characterized in terms of Reading’s zipping of posets, which is a variation of edge contractions in simplicial complexes.

1. INTRODUCTION

This work was originally motivated by uniform homotopy theory (see [50], which depends heavily on Chapter 2 of the present paper) and, independently, embedding theory (see [49]). In short, the applications force a rethink of PL topology, which involves a shift of focus from simplicial complexes to general posets (also known as van Kampen’s star complexes or Akin’s cone complexes) and cell complexes (also known as regular PL CW-posets), as well as from spaces to maps.

It gradually became clear to the author that the same kind of techniques should also apply to improve his understanding of PL transversality and collapsing. This is essentially the subject of Chapters 3 and 4. The results about collapsing may sound more convincing, yet they are largely due to a progress with transversality — even if it is quite modest per se.

A fifth motivation is the search of purely combinatorial foundations of PL topology, like in the early works by Alexander and Newman (see [46], [33; Chapter II]), but which would be (at least) as efficient as the standard foundations by Whitehead and Zeeman [66], [67], [62], [57], based on arbitrary linear subdivisions (and hence on affine geometry). A well-known obstacle here is Alexander’s problem (see [46; p. 311] and [51; comments to Theorem B] for a recent discussion): given two PL homeomorphic simplicial complexes, do they have isomorphic stellar subdivisions? (Equivalently, is having isomorphic stellar subdivisions an equivalence relation?)

Apparently it would be good enough to prove a theorem of this kind about not necessarily stellar subdivisions of not necessarily simplicial complexes, provided that all the

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notions involved are purely combinatorial. This suggests analyzing the combinatorial content of PL transversality theorems, and getting one's hands dirty with exploring more general subdivisions of more general complexes. That sounds pretty close to what the present paper is about.

The discussion of some highlights of the paper just below and in §1.A is followed by an actual introduction in §1.C, which is supposed to communicate some geometric intuition for posets and explain why they can (and perhaps should) replace simplicial complexes in a combinatorially biased treatment of PL topology.

1.A. Collapsible polyhedra and maps

It is well known and easy to see that a simplicial complex, and more generally a cell complex (=finite CW-complex whose attaching maps are PL embeddings) can be reconstructed from the poset of its *nonempty* faces (cf. [48], [14] and §1.C below). We will therefore identify simplicial and cell complexes with their posets of nonempty faces.

1.1. Constructible poset. We call a poset P *constructible* if either P has a greatest element or $P = Q \cup R$, where Q and R are order ideals (that is, if $p \leq q$ where $q \in Q$ then $p \in Q$; and similarly for R), each of Q , R and $Q \cap R$ is constructible, and every maximal element of $Q \cap R$ is covered by a maximal element of Q and by a maximal element of R .

Constructible simplicial and polytopal complexes originate in the work of M. Hochster on Cohen–Macaulay rings [40] and have been widely studied in Topological Combinatorics (see references in [37], [10]). The above definition is supposed to be the topologist's version of the notion, and has two somewhat different relations with the original combinatorialist's definition:

- If P is the “face poset”, in the combinatorialist's sense (that is, including the empty face as well!) of a simplicial or polytopal complex K , then P is constructible in the above sense if and only if K is constructible in the sense of Hochster (see Lemma 4.4).
- If P is (the poset of nonempty faces of) an *acyclic* simplicial or polytopal complex, then P is constructible in the above sense if and only if it is constructible in the sense of Hochster (see Lemma 4.5(c)).

Order complexes of constructible posets are clearly contractible (by Mayer–Vietoris and Seifert–van Kampen). The dunce hat of Zeeman [68] (also known as the Borsuk tube), that is, the mapping cone of the map $\varphi: S^1 \rightarrow S^1$ corresponding to the word $aa^{-1}a$, is an example of a contractible polyhedron that cannot be triangulated by a constructible simplicial complex [37], [38].

There is no simple relation between collapsible polyhedra and constructible simplicial complexes. Not all collapsible polyhedra can be triangulated by constructible simplicial complexes, since a constructible simplicial complex is easily seen to be pure (i.e. its maximal simplices all have the same dimension). Conversely, Hachimori constructed a

constructible simplicial complex triangulating a non-collapsible polyhedron P [37], [38]. Namely, P the mapping cone of the map $\tilde{\varphi}$ defined by the pullback diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\tilde{\varphi}} & S^1 \\ \downarrow & & \downarrow p \\ S^1 & \xrightarrow{\varphi} & S^1, \end{array}$$

where p is the double covering and φ is as above.

Theorem 1.2. *Let X be a compact polyhedron. The following are equivalent:*

- (i) X is collapsible;
- (ii) X can be triangulated by a simplicial complex whose dual poset is constructible;
- (iii) X can be cellulated by a cell complex whose dual poset is constructible.

The characterization of collapsible polyhedra by constructible posets that are dual to cell complexes may be seen as surprising. In a collapse, one bites off a single simplex, or ball, at each step; in the destruction of a constructible poset, one cuts into two parts at each step, where each part (as well as their common boundary) may be arbitrarily complex in the sense of the number of further cuts.

A PL map is called *collapsible* if its point-inverses are collapsible (so in particular nonempty). The proof of Theorem 1.2 also yields a version for maps (see Corollary 4.48 and Theorem 4.13):

Theorem 1.3. *A PL map between compact polyhedra is collapsible if and only if it can be triangulated by a simplicial map whose point-inverses have constructible dual posets.*

Note that the point-inverses of a simplicial map (other than the preimages of the vertices) are usually not simplicial complexes. It is not hard to see, however, that they are cubosimplicial complexes (Lemma 2.42).

1.4. Edge-zipping. An *edge contraction* is a surjective simplicial map $f: K \rightarrow L$ between simplicial complexes such that every vertex of L has only one preimage, apart from one vertex, whose preimage is an edge, $v * w$ (written here as the join of its vertices). We call f an *elementary edge-zipping* if any of the following equivalent conditions hold:

- $\text{lk}(v) \cap \text{lk}(w) = \text{lk}(v * w)$;
- $v * w$ is not contained in any “missing face” of K , i.e. in an isomorphic copy of $\partial\Delta^n$ in K that does not extend to an isomorphic copy of Δ^n in K ;
- the PL map triangulated by f is collapsible.

We say that a simplicial complex K *edge-zips* onto a simplicial complex L if there exists a sequence of elementary edge-zippings $K = K_0 \rightarrow \cdots \rightarrow K_n = L$.

Every triangulation of S^2 edge-zips onto $\partial\Delta^3$ (Steinitz, 1934; see [71], [65]). In fact, if K is a 2-dimensional simplicial complex, then every simplicial subdivision of K edge-zips onto K [49]. On the other hand, there exists a simplicial subdivision of $\partial\Delta^4$ that does not edge-zip onto $\partial\Delta^4$ (see [54; Example 6.1], where a stronger assertion is proved).

Proposition 1.5. *A compact polyhedron is collapsible if and only if it can be triangulated by a simplicial complex that edge-zips onto a point.*

It should not be hard to prove Proposition 1.5 directly (see Remark 4.50 for a sketch of proof of an assertion that contains the “if” implication). We will see in a moment that it follows from a more interesting result.

1.6. Zipping. Let P be a poset. Suppose that a $p \in P$ covers two incomparable elements $q, r \in P$ so that:

- if $s < p$ and $s \neq q, r$, then $s < q$ and $s < r$, and
- if $s > q$ and $s > r$, then $s \geq p$.

In this situation P is said to *elementarily zip* onto the quotient of P by the subposet $\{p, q, r\}$. (See 2.60 concerning quotient posets.) A *zipping* of posets is a sequence of elementary zippings.

This definition was introduced by N. Reading [56].

As observed in [53; 2.5(2)], if a poset P elementarily zips onto a poset Q , then the order complex of P edge-zips onto that of Q in two steps. The two edge contractions of course correspond to the chains $\{q < p\}$ and $\{r < p\}$.

Conversely:

Lemma 1.7. *If a simplicial complex K edge-zips onto a simplicial complex L , then K zips onto L .*

The author learned from E. Nevo that he had independently been aware of this fact.

Proof. It suffices to consider the case where L is obtained from K by an elementary edge-zipping; let $a * b$ be the edge being contracted. Let A_1, \dots, A_n be the simplices of the link of $a * b$ in K arranged in an order of increasing dimension, and let L_k be the subcomplex $A_1 \cup \dots \cup A_{k-1}$ of K . Let $f_k: L_k * a * b \rightarrow L_k * c$ be the simplicial quotient map identifying a with b , and let Q_k be the adjunction poset $K \cup_{f_k} L_k * c$ (see Lemma 2.59). Then $Q_1 = K/(a * b)$ (the quotient poset), so K zips onto Q_1 . Furthermore, for each $k = 1, \dots, n$ the image of the simplex $A_k * a * b$ of K in Q_k is isomorphic to $(\partial(A_k * c)) + (a * b)$, where the prejoin $X + Y$ of the posets X, Y consists of all the elements of Y placed above all the elements of X , retaining the original orders within X and Y (see §2.C). It is easy to check by induction that the image of $A_k * a * b$ is the only cell of Q_k that contains both the images of $A_k * a$ and $A_k * b$, and it follows that Q_k elementarily zips onto Q_{k+1} . Thus K zips onto $Q_n = L$. \square

As promised, Proposition 1.5 is now a consequence of Lemma 1.7, Theorem 1.2 and the following

Theorem 1.8. *A cell complex K zips onto a point if and only if the dual poset K^* is constructible.*

One can interpret Theorems 1.2 and 1.8 as providing two quite different combinatorial characterizations of collapsibility, which only magically turn out to be equivalent combinatorially and not just topologically.

Given the high flexibility of constructible posets and the very concrete character of the zipping operation, one is tempted to think that it is constructible posets and zipping that are the most adequate combinatorial representation of collapsible polyhedra — rather than, say, simplicially collapsible or edge-contractible simplicial complexes.

1.9. Scheme of the proof. The proof of Theorems 1.2 and 1.8 in Chapter 4 involves the following steps.

- If K is a triangulation of a collapsible polyhedron, then some simplicial subdivision of K is simplicially collapsible (Whitehead [66]; see also [67], [57], [3]).
- If a simplicial complex K is simplicially collapsible, then trivially it is collapsible as a poset (see definition in 4.28).
- If a poset P is collapsible, then $(P^b)^*$ is shellable (Corollary 4.44), where P^b denotes the barycentric subdivision (=the poset of nonempty faces of the order complex) and Q^* denotes the dual poset. (This step is a combinatorial abstraction of a rather well-known construction in PL topology.)
- If a poset P is shellable (see definition in 4.30), then trivially it is constructible.
- If K is a simplicial complex such that K^* is constructible, then K zips onto a point (Theorem 4.22(b)).
- If K or K^* is a cell complex, and K zips onto a point, then K transversely zips onto a point (Theorem 4.19).
- If K or K^* is a cell complex, and K transversely zips onto a point (see definition in 4.15), then K^* is transversely constructible (Theorem 4.21(a)).
- If K is transversely constructible (see definition in 4.1), then the polyhedron triangulated by K is collapsible (Lemma 4.25).

1.B. PL basics and PL transversality revisited

Having discussed the results of Chapter 4, let us now review the results of Chapters 2 and 3.

Chapter 2 develops a purely combinatorial and self-contained theory, without any homeomorphisms hidden in the background. Most results of this chapter are valid for posets of any cardinality, and serve as lemmas either for further chapters or for further papers by the author (or both).

We start by developing a combinatorial star/link/join/prejoin technique (§2.C) which enables one to reduce some basic PL homeomorphisms such as $X * Y = CX \times Y \cup_{X \times Y} X \times CY$ and $\text{lk}((x, y), X \times Y) \cong \text{lk}(x, X) * \text{lk}(y, Y)$ to combinatorial isomorphisms (which is not really possible within the usual simplicial theory).

This is complemented by a technique of canonical subdivision (§2.D), which can be seen as a nearly universal alternative to the familiar barycentric subdivision — so that, for instance, it provides an alternative combinatorial handle decomposition of a triangulated manifold (§2.F). We find the canonical subdivision to be useful in PL transversality (§3.H) and absolutely indispensable in the theory of uniform polyhedra ([50]), where the barycentric subdivision is totally useless.

We also show equivalence of a local and a global definition of (possibly locally infinite) simplicial complexes (Theorem 2.28) and compare two kinds of combinatorial weakly infinite-dimensional cubes using a version of the Davis–Januszkiewicz/Babson–Billera–Chan mirroring construction (2.53 and Theorem 2.52).

Finally, we include a combinatorial treatment of Hatcher’s homotopy colimit decomposition of simplicial maps (Theorem 2.73), which incidentally turns out to have a cubical character (Lemma 2.46), and observe that it implies Homma’s theorem on factorization of simplicial maps (Corollary 2.74).

1.10. Convention/disclaimer. Starting with Chapter 3, we confine our attention to finite posets. Additional related conventions are stated at the beginning of Chapter 3.

Starting with Chapter 3, we also occasionally use known results from PL topology. Thus the remaining chapters are not entirely self-contained. We do try to keep external references to a minimum.

Two key notions are introduced in Chapter 3: a subdivision map and a stratification map.

A special case of subdivision maps, where the domain and the range are cell complexes were studied by N. Mnëv, who calls them assembly or aggregation maps [51]. The opposite case, where the domain and the range are dual posets of simplicial complexes was considered by E. Akin [6]. The equivalence of a local and a global definition of a subdivision map (Theorem 3.4) yields a simple proof of some of Akin’s results.

In the case where the range is a cell complex, stratification maps are a combinatorial abstraction of the notion of a polyhedral mock bundle of Buoncrisiano, Rourke and Sanderson [18]. A characterization of stratification maps in terms of subdivision maps is given in Theorem 3.25.

We give a combinatorial proof of Whitehead’s collaring theorem (Theorem 3.19) and extend it to a “multi-collaring” theorem somewhat reminiscent of Grothendieck’s reflections on “tame topology” (Theorem 3.66). The multi-collaring theorem implies that a stratification map gives rise to a PL variety filtration in the sense of D. Stone [64], and in particular to a locally conelike TOP stratified set in the sense of Siebenmann [60].

We give combinatorial proofs of McCrory’s characterization of PL manifolds (Theorem 3.46), M. M. Cohen’s partition theorem for simplicial maps (Theorem 3.56) and the Buoncrisiano–Rourke–Sanderson amalgamation theorem (Theorem 3.60). The partition and amalgamation theorems — along with a subdivision theorem which we don’t consider in this paper — form the basis of the geometric treatment of generalized homology and cohomology theories (Buoncrisiano–Rourke–Sanderson, [18; Chapter II]) and are the main ingredients of the PL transversality theorem (see [18; Chapter II]).

The Buoncrisiano–Rourke–Sanderson definition of PL transversality makes sense for a pair of PL maps from polyhedra to a polyhedron [18; pp. 23–24, 35]. McCrory has shown that PL transversality of a pair of embedded polyhedra in a manifold is a symmetric relation [48]. We show that McCrory’s symmetric definition of transversality for

embeddings does not extend to the case of maps (Example 3.69); however, it does extend to a symmetric notion of “semi-transversality” of maps such that semi-transverse maps can be made transverse by a canonical procedure, involving no arbitrary choices (Corollary 3.71 and Theorem 3.72).

1.C. Geometry of posets and preposets

1.11. Face poset of a convex polytope. Let P be a convex polytope, that is, the convex hull of a finite subset of some Euclidean space \mathbb{R}^d . The relation of inclusion on the set \mathcal{F}_P of all nonempty faces of P is a partial order, so we have the *face poset* $F_P = (\mathcal{F}_P, \subset)$. If two convex polytopes are affinely equivalent, clearly their face posets are isomorphic.

If σ is an affine simplex, that is, the convex hull of a nonempty finite affinely independent set of points $S \subset \mathbb{R}^d$, then F_σ is isomorphic to the poset Δ^S of all nonempty subsets of S ordered by inclusion.

If π is a parallelepiped, that is, the convex hull of the sums of vectors in all subsets of a nonempty finite linearly independent set of vectors $S \subset \mathbb{R}^d$, then F_π is isomorphic to the poset I^S of all intervals $[S_-, S_+] = \{T \subset S \mid S_- \subset T \subset S_+\}$, where $S_+ \subset S_- \subset S$, ordered by inclusion.

1.12. Duality in polytopes. (See [36], [70] for proofs of all assertions.) If $P \subset \mathbb{R}^d$ is a convex polytope whose interior contains the origin of \mathbb{R}^d , then its *polar* $P^\star = \{x \in \mathbb{R}^d \mid (x, y) \leq 1 \text{ for all } y \in P\}$ is a convex polytope containing the origin, and $(P^\star)^\star = P$. The polar of an affine n -simplex is an affine n -simplex, and that of an n -parallelepiped is an n -cross-polytope; the other polar pairs of regular polytopes are: icosahedron/dodecahedron; 24-cell/24-cell; and 120-cell/600-cell.

The *boundary face poset* $\partial F_P = (\mathcal{F}_P \setminus \{P\}, \subset)$ of a convex polytope P whose interior contains the origin is dual to the boundary face poset ∂F_{P^\star} of the polar P^\star , in the sense that there is a bijection $\mathcal{F}_P \setminus \{P\} \rightarrow \mathcal{F}_{P^\star} \setminus \{P^\star\}$ that reverses the order by inclusion. In particular, the poset $\partial \Delta^S$ of all nonempty proper subsets of S ordered by inclusion is self-dual.

A convex polytope is called *simplicial* if every its proper face is an affine simplex. Clearly, this is equivalent to saying that the face poset of every proper face is isomorphic to a combinatorial simplex Δ^S . A convex d -polytope P is called *simple* if every its vertex is incident to precisely d edges (which is the minimal possible number). This is equivalent to saying that the poset of faces properly containing any given vertex of P is isomorphic to a combinatorial simplex. Thus a convex polytope P is simple if and only if P^\star is simplicial.

1.13. Affine simplicial complex. An *affine simplicial complex* is a finite set \mathcal{K} of non-empty affine simplices in some \mathbb{R}^d such that

- (i) if $\sigma \in \mathcal{K}$ and τ is a non-empty face of σ , then $\tau \in \mathcal{K}$;
- (ii) if $\sigma, \tau \in \mathcal{K}$, then $\sigma \cap \tau \in \mathcal{K}$.

The simplices of \mathcal{K} form the *face poset* $K = (\mathcal{K}, \subset)$ with respect to inclusion.

An affine simplicial complex is called *flag* if every subcomplex of K whose face poset is isomorphic to $\partial\Delta^S$, $|S| > 2$, is the boundary of some simplex $\sigma \in \mathcal{K}$.

1.14. Affine cone complex. An *affine cone complex* is a family \mathcal{P} of subcomplexes of an affine simplicial complex \mathcal{K} such that

- (i) for each $\sigma \in \mathcal{P}$, all maximal simplices of σ (i.e. those simplices of σ that are not faces of other simplices of σ) share a common vertex, denoted $\hat{\sigma}$;
- (ii) for each $\sigma \in \mathcal{P}$, the set $\partial\sigma$ of all simplices of σ disjoint from $\hat{\sigma}$ is a union of elements of \mathcal{P} ;
- (iii) if $\sigma, \tau \in \mathcal{P}$ and $\hat{\sigma} = \hat{\tau}$, then $\sigma = \tau$.

The cones (i.e. the elements) of \mathcal{P} form its *face poset* $P = (\mathcal{P}, \subset)$ with respect to inclusion.

Affine cone complexes arose in van Kampen's dissertation [43] and also in the work of M. M. Cohen and E. Akin (see [5; p. 456]) and were further studied by McCrory [48]. The following proposition is found in [48]:

Proposition 1.15. *Every finite poset is isomorphic to the poset of cones of some affine cone complex.*

Proof. Given a finite poset $P = (\mathcal{P}, \leq)$, the combinatorial simplex $\Delta^{\mathcal{P}}$ is the face poset of an affine simplex σ whose vertices may be identified with the elements of \mathcal{P} . Every nonempty finite chain $p_1 < \dots < p_n$ in P corresponds to the affine subsimplex $p_1 * \dots * p_n$ of σ with vertices p_1, \dots, p_n . The set of all such subsimplices forms an affine simplicial subcomplex $\Delta(P)$ of σ . For each $p \in P$ let $\lceil p \rceil$ be the subposet of P consisting of all $q \leq p$. Then $\Delta(\lceil p \rceil)$ is an affine simplicial subcomplex of $\Delta(P)$. It is easy to see that $\{\Delta(\lceil p \rceil) \mid p \in P\}$ is an affine cone complex whose face poset is isomorphic to P . \square

We find it instructive to review the proof of a special case in geometric terms.

Corollary 1.16. *The face poset of every convex polytope is isomorphic to the face poset of some affine cone complex.*

Proof. If P is a convex polytope in \mathbb{R}^d , then there exists an affine simplicial complex P^b in \mathbb{R}^d such that each face of P is a union of simplices of P^b . Namely, P^b has one vertex $\hat{\sigma}$ in the interior of each nonempty face σ of P , and one simplex for every nonempty chain $\sigma_1 \subset \dots \subset \sigma_n$ of faces of P , namely the affine simplex $\hat{\sigma}_1 * \dots * \hat{\sigma}_n$ with vertices $\hat{\sigma}_1, \dots, \hat{\sigma}_n$. Now the family of all subcomplexes of P^b triangulating the nonempty faces of P is an affine cone complex whose face poset is isomorphic to F_P . \square

1.17. Dual cone complex of a simplicial complex. Given an affine simplicial complex \mathcal{K} , by Proposition 1.15 its face poset $K = (\mathcal{K}, \leq)$ is isomorphic to the face poset of an affine cone complex \mathcal{P} , consisting of subcomplexes of an affine simplicial complex \mathcal{K}^b . The latter can be constructed similarly to the proof of Corollary 1.16, so that simplices of K are unions of simplices of K^b . The dual poset $K^* = (\mathcal{K}, \geq)$ has the

same chains as K , and it follows that it is isomorphic to the face poset of the affine cone complex \mathcal{Q} , consisting of subcomplexes of \mathcal{K}^b of the form $\sigma^* = \{\hat{\sigma}_1 * \cdots * \hat{\sigma}_n \mid \sigma_1 > \cdots > \sigma_n \geq \sigma\}$, where $\sigma \in K$.

1.18. Coboundary of a simplicial cochain. Let \mathcal{K} be an affine simplicial complex, and let \mathcal{Q} be its dual cone complex, as above. The simplicial cochain $c_\sigma \in C^k(\mathcal{K}; \mathbb{Z}/2)$ given by the indicator function of a k -simplex $\sigma \in \mathcal{K}$ has coboundary $\delta(c_\sigma) = \sum_i c_{\tau_i} \in C^{k+1}(\mathcal{K}; \mathbb{Z}/2)$, where $\partial(\sigma^*) = \bigcup_i \tau_i^*$ in \mathcal{Q} ; see [28; figure on p. 26].

Proposition 1.19. *Let \mathcal{P} be an affine cone complex.*

- (a) *If $\sigma \in \mathcal{P}$, then every vertex of σ is $\hat{\rho}$ for some $\rho \in \mathcal{P}$ with $\rho = \sigma$ or $\rho \subset \partial\sigma$.*
- (b) *For all $\sigma, \tau \in \mathcal{P}$, either $\sigma = \tau$, or $\sigma \subset \partial\tau$, or $\tau \subset \partial\sigma$, or else $\sigma \cap \tau \subset \partial\sigma \cap \partial\tau$.*
- (c) *For all $\sigma, \tau \in \mathcal{P}$, the intersection $\sigma \cap \tau$ is a union of cones of \mathcal{P} .*

This can be easily deduced from Proposition 1.15, but a direct proof contributes to our understanding of affine cone complexes.

Proof. (a). If $v \neq \hat{\sigma}$, then $v \in \partial\sigma$. By (ii), there exists a cone ρ of \mathcal{P} contained in $\partial\sigma$ and containing v . By finiteness we may assume that ρ contains no other cone with this property. Then $v \notin \partial\rho$ by (ii). Hence $v = \hat{\rho}$. \square

(b). We will show that either $\tau = \sigma$, or $\tau \subset \partial\sigma$, or else $\sigma \cap \tau \subset \partial\tau$. If $\hat{\tau} = \hat{\sigma}$, then $\tau = \sigma$ by (iii). If $\hat{\tau} \neq \hat{\sigma}$ and $\hat{\tau} \in \sigma$, then by (a), $\hat{\tau} = \hat{\rho}$ for some $\rho \in \mathcal{P}$, $\rho \subset \partial\sigma$. Then $\tau = \rho$ by (iii), hence $\tau \subset \partial\sigma$. Finally, if $\hat{\tau} \notin \sigma$, then $\sigma \cap \tau \subset \partial\tau$. \square

(c). By (b) we may assume that $\sigma \cap \tau \subset \partial\sigma \cap \partial\tau$. Let A be a simplex of $\sigma \cap \tau$. Similarly to the proof of (a), A is contained in some cone λ of \mathcal{P} contained in $\partial\sigma$ such that $A \notin \partial\lambda$, and in some cone μ of \mathcal{P} contained in $\partial\tau$ such that $A \notin \partial\mu$. Then by (b), $\lambda = \mu$. \square

1.20. Flag affine Δ -set. An *affine Δ -set* is an affine simplicial complex \mathcal{K} where every simplex is endowed with a total ordering of vertices so that every inclusion between simplices of \mathcal{K} is monotone on the sets of vertices. (See [31], [58] concerning general Δ -sets.)

Since the total orderings of vertex sets of simplices agree, they extend to a partial order on the set of all vertices of \mathcal{K} . Thus an affine Δ -set is simply an affine simplicial complex endowed with a partial ordering of vertices such that every two vertices connected by an edge are comparable.

We call an affine Δ -set $\mathcal{L} = (\mathcal{K}, \leq)$ *flag* if \mathcal{K} is a flag simplicial complex.

If \mathcal{P} is an affine cone complex consisting of subcomplexes of an affine simplicial complex \mathcal{K} , then a partial ordering of the vertices of \mathcal{K} is given by the relation $\hat{\sigma} \leq \hat{\tau}$ if and only if $\sigma \subset \tau$. Clearly, two vertices of \mathcal{K} are comparable if and *only if* they are connected by an edge. Moreover, it follows from Proposition 1.19(b) that \mathcal{K} is a flag complex. Thus affine cone complexes can be viewed as a special case of flag affine Δ -sets.

1.21. Strictly acyclic relation. If $\mathcal{L} = (\mathcal{K}, \leq)$ is a flag affine Δ -set, then the 1-skeleton of \mathcal{K} carries the structure of an acyclic digraph (=a directed graph with no directed

cycles). An acyclic digraph $G = (V, E)$ amounts to a binary relation $E \subset V \times V$ that is *strictly acyclic* in the sense that it admits no cycles of the form $v_1 E v_2 E \dots E v_n E v_1$. Conversely, an acyclic digraph embedded in general position in \mathbb{R}^d spans a flag affine simplicial complex in \mathbb{R}^d , whose vertices are partially ordered by the relation: $v \leq w$ if and only if there exists a directed path from v to w in the 1-skeleton.

Proposition 1.22. *Every partial order on a finite set extends to a total order.*

This is well-known, but we would like to emphasize a particular canonical extension.

Proof. A finite poset $P = (\mathcal{P}, \leq)$ admits a monotone embedding in the combinatorial simplex $\Delta^{\mathcal{P}}$ via $p \mapsto [p]$, where $[p] = \{q \in \mathcal{P} \mid q \leq p\}$. It remains to compose this embedding with the monotone surjection $\Delta^{\mathcal{P}} \rightarrow \{1, 2, \dots, |\mathcal{P}|\}$, $S \mapsto |S|$. \square

Proposition 1.22 relates flag affine Δ -sets to PL Morse theory, cf. [12] and [48; Example on p. 275]. See also Kearton–Lickorish [44] (along with references to Kosiński and Kuiper therein), Forman [30] (along with elaborations in [55] and [45]) and Bestvina [11].

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2. FINITE AND INFINITE POSETS

2.A. Basic notions

2.1. Posets and anti-reflexive relations. If \prec is an *anti-reflexive* relation on a set \mathcal{P} (i.e. $x \not\prec x$ for all $x \in \mathcal{P}$), its *inclusive* counterpart \preceq is defined by $x \preceq y$ iff $x \prec y$ or $x = y$; it is reflexive (i.e. $x \preceq x$ for all $x \in \mathcal{P}$). For a reflexive relation \preceq , its *exclusive* counterpart \prec is defined by $x \prec y$ iff $x \preceq y$ and $x \neq y$; it is anti-reflexive. The operations of inclusive/exclusive counterpart constitute mutually inverse bijections between the set of all reflexive relations on a set \mathcal{P} and the set of all anti-reflexive relations on \mathcal{P} .

A binary relation $<$ on a set \mathcal{P} is called a *strict partial order* if it is anti-reflexive and *transitive* (i.e. $x < y$ and $y < z$ imply $x < z$ for all $x, y, z \in \mathcal{P}$). It is easy to see that $<$ is a strict partial order iff its inclusive counterpart \leq is a *partial order*, that is, is reflexive, *anti-symmetric* (i.e. $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in \mathcal{P}$) and transitive. A set endowed with a strict or non-strict partial order is called a *poset*.

2.2. Preposets and acyclic relations. By a *preposet* we mean a set \mathcal{P} endowed with a *strictly acyclic* relation, in the sense that there exists no sequence $x_0, \dots, x_n \in \mathcal{P}$, with n being a nonnegative integer, such that $x_0 \prec x_1 \prec \dots \prec x_n \prec x_0$. In particular, a strictly acyclic relation is anti-reflexive. We also write $x \prec y$ as $y \succ x$.

The inclusive counterpart \preceq of a strictly acyclic relation \prec is characterized by being reflexive and *acyclic* in the sense that if $x_0, \dots, x_n \in \mathcal{P}$, with n being a nonnegative

integer, satisfy $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_0$, then $x_0 = \cdots = x_n$. In particular, \preceq is anti-symmetric. By the above, a preposet may be equivalently viewed as a set \mathcal{P} endowed with a binary relation \preceq that is reflexive and acyclic.

2.3. Transitive closure and covering relation. The *transitive closure* of a binary relation \prec on a set \mathcal{P} is the relation $\prec\prec$ on \mathcal{P} defined by $x \prec\prec y$ iff there exist $z_1, \dots, z_n \in \mathcal{P}$ for some nonnegative integer n such that $x \prec z_1 \prec \cdots \prec z_n \prec y$; it is clearly transitive.

It is easy to see that a binary relation is (strictly) acyclic iff its transitive closure is a (strict) partial order. Thus every poset is a preposet, and for every preposet $P = (\mathcal{P}, \prec)$, its transitive closure $\langle P \rangle := (\mathcal{P}, \prec\prec)$ is a poset. So one may view preposets not just as a generalization of posets, but more specifically as posets endowed with an additional structure.

Given a poset $P = (\mathcal{P}, \leq)$, one defines the *covering* relation \prec on P by $p \prec q$ if $p < q$ and there exists no $r \in \mathcal{P}$ with $p < r < q$. Clearly, \prec is the minimal strictly acyclic relation whose transitive closure is $<$.

2.4. Monotone maps. Let $P = (\mathcal{P}, \leq)$ and $Q = (\mathcal{Q}, \preceq)$ be [pre]posets. An order preserving or *monotone* map between them is a map $f: \mathcal{P} \rightarrow \mathcal{Q}$ such that $v \leq w$ implies $f(v) \preceq f(w)$ for all $v, w \in \mathcal{P}$. It is called a *monotone embedding* if the converse implication holds as well. Every monotone embedding is obviously injective, but not every injective monotone map is a monotone embedding. An *isomorphism* of [pre]posets is a monotone bijection whose inverse is monotone, or equivalently a surjective monotone embedding.

We say that Q is a *sub[pre]poset* of P , and write $Q \subset P$, if \mathcal{Q} is a subset of \mathcal{P} and the inclusion $\mathcal{Q} \hookrightarrow \mathcal{P}$ is an embedding of Q into P .

The *dual* of a [pre]poset $P = (\mathcal{P}, \leq)$ is the [pre]poset $P^* := (\mathcal{P}, \geq)$.

From now on we often do not distinguish between a [pre]poset $P = (\mathcal{P}, \preceq)$ and its underlying set \mathcal{P} (by an abuse of notation).

2.5. Unary operations: C , C^* , ∂ , ∂^* . Let P be a [pre]poset. The *cone* CP over P is obtained by adjoining to P an additional element, denoted $\hat{1}$ (or $+\infty$), which is set to be greater than every element of P . The *dual cone* $C^*P := (C(P^*))^*$ is obtained by adjoining to P an additional element, denoted $\hat{0}$ (or $-\infty$), which is set to be less than every element of P .

If P is a [pre]poset, its *boundary* ∂P is the sub[pre]poset of P consisting of all $p \in P$ such that $p \leq q$ for some $q \in P$ such that there exists precisely one element $r \in P$ satisfying $r > q$. The *coboundary* $\partial^*P = (\partial(P^*))^*$. We will later see that ∂ is related to the boundary of a manifold; on the other hand, ∂^* can be seen to be related to the coboundary of a cochain (see 1.18). Note that $\partial(CP) = P = \partial^*(C^*P)$.

Example 2.6 (2^S and Δ^S). Let S be a set (possibly infinite). The relation of inclusion on the set 2^S of all subsets of S is a partial order. The resulting *subset poset* $2^S = (2^S, \subset)$ is isomorphic to its own dual (by taking the complement). The poset Δ^S of all nonempty subsets of S will be called a (*combinatorial*) *simplex* or the *S-simplex*, or the *n-simplex*

(notation: Δ^n) in the case where S is $[n+1] := \{0, 1, \dots, n\}$. If $T \subset S$ is non-empty, Δ^T is called a *face* of Δ^S . Faces that are 0-simplices (i.e. singletons) are also called *vertices*, and faces that are 1-simplices are also called *edges*. Note that $\partial\Delta^S = \partial(\partial^*2^S)$ is isomorphic to its own dual (compare Example 1.12).

2.7. Cones. Let P be a [pre]poset. The *cone* $\lceil p \rceil$ (resp. the *dual cone* $\lfloor p \rfloor$) of a $p \in P$ is the sub[pre]poset of P consisting of all $q \in P$ satisfying $q \leq p$ (resp. $q \geq p$). We may also write $\lceil p \rceil_P$ and $\lfloor p \rfloor^P$ to emphasize the [pre]poset P .

The definitions of $\lceil p \rceil$ and $\lfloor p \rfloor$ are in agreement with the previously defined cone and dual cone over a poset; namely, $\lceil p \rceil$ is the cone over $\partial\lceil p \rceil$, and $\lfloor p \rfloor$ is the dual cone over $\partial^*\lfloor p \rfloor$. The duality is expressed by $\lfloor p \rfloor = \lceil p^* \rceil^*$. The notation¹ $\lfloor q \rfloor$ and $\lceil q \rceil$ can be thought of as just a concise form of the interval notation $[p, +\infty)$ and $(-\infty, p]$, where the square brackets undergo a counterclockwise 90° rotation.

Remark 2.8. If \mathcal{P} is an affine simplicial complex and \preceq is the inclusion relation (see Example 1.13), the cone of an affine simplex $\sigma \in \mathcal{P}$ is the face poset F_σ , viewed as the poset of all simplices of the subcomplex of \mathcal{P} triangulating σ (in fact, this subcomplex does happen to be a “cone” in the terminology of Rourke–Sanderson [57; 2.8(7)]); whereas the dual cone of σ is isomorphic to what is known as the “dual cone” of σ in PL topology (see [57; 2.27(6)]).

2.9. Cone complexes. By a *cone complex* we mean a countable poset where every cone is finite. A *cone precomplex* is a preposet whose transitive closure is a cone complex. A cone [pre]complex P such that the dual [pre]poset P^* is also a cone [pre]complex is called *locally finite*.

Remark 2.10. Cone precomplexes other than cone complexes arise in practice as triple deleted prejoins of cone complexes and as mapping cylinders of non-closed monotone maps between cone complexes; the non-closed monotone maps in turn arise in practice as diagonal maps $P \rightarrow P \times P$ and more importantly as bonding maps between nerves of coverings.

Lemma 2.11. (a) *Every preposet admits a monotonous injection into a simplex.*

(b) *A preposet is a poset iff it is isomorphic to a subposet of a simplex.*

Proof. (b). Since every simplex is itself a poset, every preposet embedded into a simplex is a poset. Conversely, every poset $P = (\mathcal{P}, \geq)$ is isomorphic to the poset of cones of P ordered by inclusion of their underlying sets, which is a subposet of $\Delta^{\mathcal{P}}$. \square

(a). This follows from (b) since the inclusion of every preposet in its transitive closure is monotonous. \square

¹These are typeset in T_EX with the usual `\lfloor`, `\rfloor`, `\lceil` and `\rceil` delimiters, downscaled using `\scriptstyle` (in regular text) and `\scriptscriptstyle` (in sub- and superscripts), or their context-sensitive combination formed with the aid of `\mathchoice`. (The `\mathsmaller` command of the `relsize` package produces similar results except in displayed equations.)

2.12. Closed and open subposets. Let P be a [pre]poset. A sub[pre]poset Q of P is called *closed* (resp. *open*) if $p \leq q$ (resp. $p \geq q$), where $q \in Q$, implies $p \in Q$. Note that if Q is a closed subpreposet of P , then $\langle Q \rangle$ is a closed subposet of $\langle P \rangle$.

The *closure* (resp. *open hull*) of a sub[pre]poset Q of P is the sub[pre]poset $\lceil Q \rceil$ (resp. $\lfloor Q \rfloor$) of P consisting of all $p \in P$ such that $p \leq q$ (resp. $p \geq q$) for some $q \in Q$. If P is a poset, $\lceil Q \rceil$ and $\lfloor Q \rfloor$ coincide with the smallest closed subposet and the smallest open subposet containing Q .

The collection of all open subposets of a poset $Q = (\mathcal{Q}, \leq)$ determines a topology on \mathcal{Q} , known as the Alexandroff or the right order topology. It goes back to P. S. Alexandroff [7] that the following are equivalent for a T_0 topological space X :

- (i) Arbitrary intersections of open sets are open in X ;
- (ii) Every point has a smallest neighborhood in X ;
- (iii) X is homeomorphic to a poset endowed with the Alexandroff topology.

2.13. Closed and open maps. It is easy to see that the following are equivalent for a map $f: P \rightarrow Q$ between posets:

- (i) f is monotone;
- (ii) f continuous with respect to the Alexandroff topologies;
- (iii) $f(\lceil p \rceil) \subset \lceil f(p) \rceil$ for each $p \in P$;
- (iv) $f(\lfloor p \rfloor) \subset \lfloor f(p) \rfloor$ for each $p \in P$.

For instance, to see that (ii) implies (i), we note that if $p' > p$, then p' lies in every open subposet of P containing p ; whereas if $f(p) \not\leq f(p')$, then $f(p')$ does not lie in the open subposet $\lfloor f(p) \rfloor$ of Q .

In general topology, a continuous map is called *open* [resp. *closed*] if it sends every open [closed] set to an open [closed] one. It is now obvious that a monotone map $f: P \rightarrow Q$ between posets is open [closed] (with respect to the Alexandroff topologies) if and only if $f(\lfloor p \rfloor) = \lfloor f(p) \rfloor$ [resp. $f(\lceil p \rceil) = \lceil f(p) \rceil$] for every $p \in P$.

Example 2.14. An example of a non-closed monotone map is given by the diagonal embedding of the 1-simplex Δ^1 into $\Delta^1 \times \Delta^1$.

2.B. Suprema, atoms and simplices

2.15. Conditionally complete poset. A poset P is *conditionally complete* if every non-empty $Q \subset P$ that has an upper bound in P (i.e. a $p \in P$ such that $Q \subset \lceil p \rceil$) also has a least upper bound in P (i.e. an upper bound $p \in P$ such that $\lfloor p \rfloor$ contains all upper bounds of Q in P).

Remark 2.16. Conditionally complete posets should not be confused with bounded complete posets (also known as consistently or coherently complete posets), where every (not necessarily non-empty!) subset that has an upper bound has a least upper bound. It is easy to see that bounded complete posets are precisely those conditionally complete posets that have a least element; and complete posets are precisely those conditionally complete posets that have a least element and a greatest element.

The following lemma is well-known, cf. [13]:

Lemma 2.17. *A poset P is conditionally complete if and only if every non-empty subset of P that has a lower bound in P also has a greatest lower bound in P .*

Proof. By symmetry, it suffices to prove the ‘only if’ assertion. If P is conditionally complete and a subset $Q \subset P$ has a lower bound in P , then the set L of all lower bounds of Q in P is nonempty and has an upper bound in P (specifically, any element of Q will do). Then there exists the greatest lower bound u of L in P , that is, $\lceil u \rceil$ contains L and $\lfloor u \rfloor$ contains all upper bounds of L , in particular, all of Q . By definition, u is the greatest lower bound of Q . \square

Corollary 2.18. (a) *Every simplex is a conditionally complete poset.*

(b) *Every closed subposet of a conditionally complete poset is conditionally complete.*

Proof. (a). The greatest lower bound of a set of A of nonempty subsets $S_\alpha \subset S$ is their intersection $\bigcap_\alpha S_\alpha$, if it is nonempty; else A has no lower bounds. \square

(b). Let P be a conditionally complete poset and Q its closed subposet. If the greatest lower bound of a subset $S \subset Q$ exists in P , then it belongs to Q , since Q contains all its lower bounds. \square

2.19. Full subposet. A subposet Q of a poset P is called *full* in P , if every cone of P meets Q in a cone of Q or in the empty set. When Q is a closed subposet, this is McCrory’s definition (see also [48; Lemma 2.6]). When P is a simplicial complex, this is the usual definition of a full subcomplex, cf. [57].

It is easy to see that open subposets are full.

Lemma 2.20. *A full subposet of a conditionally complete poset is conditionally complete.*

Proof. Let F be a full closed subposet of a conditionally complete poset P . Given a subset $S \subset F$, the set T of all lower bounds of S in P is a cone since P is conditionally complete. Since F is a subposet of P , the set of all lower bounds of S in F equals $T \cap F$, which is a cone since F is full in P . \square

Lemma 2.21. *Let P be a poset that is a union of closed subposets Q and R such that $Q \cap R$ is full in Q and in R . If Q and R are conditionally complete, then so is P .*

Proof. Suppose that S is a nonempty subset of P , and $p \in P$ is an upper bound of S . By symmetry we may assume that $p \in Q$. Since Q is closed, $S \subset Q$. Since Q is conditionally complete, S has a least upper bound q in Q . If $r \notin Q$ is an upper bound of S in P , then $r \in R$. In this case $S \subset Q \cap R$. Since $Q \cap R$ is full in Q , we get that S has an upper bound $s \leq r$ in $Q \cap R$. Since q is the least upper bound of S in Q , we have $q \leq s$. Hence $q \leq r$, which shows that q is a least upper bound of S in P . \square

Lemma 2.22. *Let Q be a full subposet of a poset P .*

(a) *A subposet R of Q is full in Q if and only if it is full in P .*

(b) *If U is an open subposet of P , then $Q \cap U$ is full in U .*

Proof. (a). The “only if” direction is obvious. Conversely, let $q \in Q$. Then $\lceil q \rceil_Q = \lceil q \rceil_P \cap Q$, so $\lceil q \rceil_Q \cap R = \lceil q \rceil_P \cap R$. \square

(b). Since U is open, $U \cap Q$ is open in Q . Then $U \cap Q$ is full in Q . Since Q is full in P , by the “only if” in (a), $U \cap Q$ is full in P . On the other hand, since U is open in P , it is full in P . Then by the “if” in (a), $U \cap Q$ is full in U . \square

Lemma 2.23. *If Q is a full subposet of a poset P , then $\lfloor Q \rfloor$ strong deformation retracts onto Q via a monotone map $r: \lfloor Q \rfloor \rightarrow Q$, homotopic to the identity by a monotone homotopy $h: \lfloor Q \rfloor \times [2] \rightarrow \lfloor Q \rfloor$ that extends the projection $Q \times [2] \rightarrow Q$.*

Here $[2] = \{1, 2\}$ with the usual order.

Proof. Define $r(p) = q$, where $\lceil p \rceil \cap Q = \lceil q \rceil$. Then $r(q) = q$ for each $q \in Q$, and $r(p) \leq p$ for each $p \in \lfloor Q \rfloor$, so $r \cup \text{id}_{\lfloor Q \rfloor}$ is the desired homotopy h . \square

2.24. Atoms. An element σ of a poset P is called an *atom* of P , if $\lceil \sigma \rceil = \{\sigma\}$. The set of all atoms of P will be denoted $A(P)$. A poset P is called *atomic*, if every its element is the least upper bound of some subset of $A(P)$.² It is easy to see that $A(\lceil \sigma \rceil) = A(P) \cap \lceil \sigma \rceil$. Hence every element σ of an atomic poset is the least upper bound of $A(\lceil \sigma \rceil)$.

In the case of atomic posets Lemma 2.11 admits an amplification:

Lemma 2.25. *If P is an atomic poset, then the formula $\sigma \mapsto A(\lceil \sigma \rceil)$ defines an embedding of P into $\Delta^{A(P)}$.*

Proof. It is easy to see that the formula defines a monotone map $f: P \rightarrow 2^{A(P)}$. Since each $\sigma \in P$ is the least upper bound of $A(\lceil \sigma \rceil)$, the latter is nonempty (whence the image of f is in $\Delta^{A(P)}$). If $A(\lceil \tau \rceil) \subset A(\lceil \sigma \rceil)$, then the least upper bound σ of $A(\lceil \sigma \rceil)$ is an upper bound of $A(\lceil \tau \rceil)$. Hence its least upper bound τ satisfies $\tau \leq \sigma$ (whence f is an embedding). \square

Lemma 2.26. *An atomic poset P is conditionally complete if and only if every non-empty $R \subset A(P)$ that has an upper bound in P has a least upper bound in P .*

Proof. Only the “if” direction needs a proof. Suppose we are given an $S \subset P$ that has an upper bound in P . Then so does $R := A(\lceil S \rceil)$. Hence by our assumption R has a least upper bound ρ . Since P is atomic, every $p \in S$ is the least upper bound of $A(\lceil p \rceil) = A(\lceil S \rceil) \cap \lceil p \rceil$. Since ρ is an upper bound of R , it is an upper bound of its subset $A(\lceil p \rceil)$ for each $p \in S$. However p is the least upper bound of the same set, so ρ is an upper bound p . Thus ρ is an upper bound of S . If ρ' is another upper bound of S , then ρ' is an upper bound of $A(\lceil S \rceil)$. But ρ is the least upper bound of the same set, so $\rho' \geq \rho$. Thus ρ is the least upper bound of S . \square

²In the literature on lattice theory such posets are called “atomistic”, whereas “atomic” has a different meaning. Our usage of the term can be found e.g. in [4].

2.27. Simplicial complex. A *simplicial* poset is a conditionally complete poset where every cone is isomorphic to a simplex. (Compare [9].) A simplicial cone complex is abbreviated to a *simplicial complex*. The cones of a simplicial complex K are thus called its *simplices*. Clearly, every simplicial complex is atomic.

Theorem 2.28. *A poset is simplicial iff it is isomorphic to a subcomplex of a simplex.*

In particular, it follows that a cone complex is a simplicial complex iff it is isomorphic to a subcomplex of a simplex.

Proof. The ‘if’ assertion is straightforward. Every cone of a simplex is a simplex. A subcomplex of a simplex is a conditionally complete poset by Corollary 2.18.

Conversely, let K be a simplicial complex. Let us consider the embedding $f: K \rightarrow \Delta^{A(K)}$ constructed in Lemma 2.25. If $T \subset A(\lceil \sigma \rceil)$, then the least upper bound σ of $A(\lceil \sigma \rceil)$ is an upper bound of T , hence its least upper bound τ exists and satisfies $\tau \leq \sigma$. Therefore $A(\lceil \tau \rceil) = A(K) \cap \lceil \tau \rceil$ contains T , moreover τ is the least upper bound of $A(\lceil \tau \rceil)$, as well as of T . Since $\lceil \tau \rceil$ is isomorphic to a simplex, this implies $T = A(\lceil \tau \rceil)$. So $f(\tau) = T$, whence the image of f is a subcomplex of $\Delta^{A(K)}$. \square

2.29. Barycentric subdivision. Let $P = (\mathcal{P}, \preceq)$ be a preposet. A *chain* in P is a $Q \subset \mathcal{P}$ that is a totally ordered by \prec (that is, for each $p, q \in Q$ either $p \prec q$ or $p \succeq q$; note that this already implies that \preceq is transitive on Q). The poset P^b of all nonempty finite chains of P ordered by inclusion is a subcomplex of $\Delta^{\mathcal{P}}$ and so a simplicial poset (a simplicial complex if P is countable); it is called the *barycentric subdivision* of P .

Clearly, if Q is a closed subposet of a poset P , then Q^b is a full subcomplex of P^b .

2.30. Flag complex. A *flag complex* is a simplicial complex K such that every subcomplex of K that is isomorphic to the boundary of a simplex of dimension > 1 is the boundary of some simplex of K . Obviously, every full subcomplex of a flag complex is a flag complex.

It is easy to see that the barycentric subdivision of every cone precomplex is a flag complex; and that if Q is embedded in P , then Q^b is a full subcomplex of P^b . Using these facts, it is easy to prove

Proposition 2.31. *Let K be a simplicial complex and L a subcomplex of K^b .*

(a) *L is a flag complex iff it is the barycentric subdivision of the image of some cone precomplex under a monotone injective map into K .*

(b) *L is a full subcomplex of K^b iff it is the barycentric subdivision of some cone complex embedded in K .*

2.32. Simplicial maps. A closed map between simplicial posets is called *simplicial*. Every map of sets $f: S \rightarrow T$ induces a simplicial map $\Delta^f: \Delta^S \rightarrow \Delta^T$. If K is a subcomplex of Δ^S and L is a subcomplex of Δ^T , it is easy to see that every simplicial map $K \rightarrow L$ is a restriction of Δ^f for some $f: S \rightarrow T$.

If $f: P \rightarrow Q$ is a monotone map between preposets, it sends every nonempty finite chain of P into a nonempty finite chain of Q . The resulting map $f^\flat: P^\flat \rightarrow Q^\flat$ is the restriction to a subcomplex of the simplicial map $\Delta^f: \Delta^P \rightarrow \Delta^Q$, so it is simplicial.

2.33. Full map. Let us call a monotone map of posets $f: P \rightarrow Q$ *full*, if $f^{-1}(q)$ is full in P for each $q \in Q$.

Lemma 2.34. *Simplicial maps are full.*

Proof. Let $f: K \rightarrow L$ be a simplicial map between simplicial complexes. Pick some $\sigma \in L$, and let $F = f^{-1}(\sigma)$. If $\tau \in K$ is such that $\lceil \tau \rceil \cap F \neq \emptyset$, then $f(\tau) \geq \sigma$. If $\tau \in F$, then $\lceil \tau \rceil \cap F$ is the cone $\lceil \tau \rceil_F$ since F is a subposet of K . So without loss of generality we may assume that $f(\tau) > \sigma$. Then $\lceil \tau \rceil = \lceil \mu \rceil * \lceil \nu \rceil$, where $f(\mu) = \sigma$ and $f(\nu) \cap \sigma = \emptyset$. Hence $\lceil \tau \rceil_K \cap F = \lceil \mu \rceil_F$. \square

2.C. Basic operations

Let $P = (\mathcal{P}, \leq)$ and $Q = (\mathcal{Q}, \leq)$ be preposets.

The *prejoin* $P + Q$ is the preposet $(\mathcal{P} \sqcup \mathcal{Q}, \preceq)$, where $p \preceq q$ iff either $p \leq q$ and both $p, q \in P$; or $p \leq q$ and both $p, q \in Q$; or $p \in P$ and $q \in Q$. Clearly, the prejoin of two posets is a poset. Note that $CP \simeq P + pt$ and $C^*P \simeq pt + P$.

The *product* $P \times Q$ is the preposet $(\mathcal{P} \times \mathcal{Q}, \preceq)$, where $(p, q) \preceq (p', q')$ iff $p \leq p'$ and $q \leq q'$. It is easy to see that $2^S \times 2^T \simeq 2^{S \sqcup T}$ naturally in S and T .

The *join* $P * Q := \partial^*(C^*P \times C^*Q)$ is obtained from $(C^*P) \times (C^*Q)$ by removing the bottom element $(\hat{0}, \hat{0})$. Thus $C^*(P * Q) \simeq C^*P \times C^*Q$, whereas $P * Q$ itself is the union $C^*P \times Q \cup P \times C^*Q$ along their common part $P \times Q$.

From the above, $\Delta^S * \Delta^T \simeq \Delta^{S \sqcup T}$ naturally in S and T . It follows that the join of simplicial complexes $K \subset \Delta^S$ and $L \subset \Delta^T$ is isomorphic to the simplicial complex $\{\sigma \cup \tau \subset S \sqcup T \mid \sigma \in K \cup \{\emptyset\}, \tau \in L \cup \{\emptyset\}, \sigma \cup \tau \neq \emptyset\} \subset \Delta^{S \sqcup T}$.

The join and the prejoin are related via barycentric subdivision: $(P + Q)^\flat \simeq P^\flat * Q^\flat$. Indeed, a nonempty finite chain in $P + Q$ consists of a finite chain in P and a finite chain in Q , at least one of which is nonempty. Note that in contrast to prejoin, join is commutative: $P * Q \simeq Q * P$. Prejoin is associative; in particular, $C(C^*P) \simeq C^*(CP)$.

Remark 2.35. In the case where P and Q are finite simplicial complexes, the above mentioned isomorphism

$$P * Q \simeq C^*P \times Q \bigcup_{P \times Q} P \times C^*Q$$

can be regarded as a combinatorial form of the well-known (cf. [62; 4.3.20]) homeomorphism

$$X * Y \cong (pt * X) \times Y \bigcup_{X \times Y} X \times (pt * Y),$$

where $X = |P|$ and $Y = |Q|$. However it does not quite fit in the familiar simplicial realm even in this case, for C^*P and $P \times Q$ are no longer simplicial complexes.

If P is a conditionally complete poset, then obviously CP and C^*P are conditionally complete. If P and Q are conditionally complete, then obviously so is $P \times Q$. In addition, $P * Q$ is conditionally complete since it is an open subposet of $C^*P \times C^*Q$.

2.36. Van Kampen duality. If P and Q are preposets, their *cojoin* $P \bar{*} Q = (P^* * Q^*)^*$. By dualizing $C^*X \times C^*Y \simeq C^*(X * Y)$, where $X = P^*$ and $Y = Q^*$, one obtains

$$CP \times CQ \simeq C(P \bar{*} Q).$$

In the case where P and Q are posets, this formula was known already to E. R. van Kampen [43], cf. [48; Proposition 1.2]. It implies, for instance, that the boundary of the n -cube is dual (as a poset) to the boundary of the n -cross-polytope (compare Example 1.12):

$$\bigstar_{i=1}^n \partial I^1 \simeq (\partial I^n)^*.$$

Lemma 2.37. $CP * CQ \simeq C \left(C^*P \bar{*} Q \cup_{P \bar{*} Q} P \bar{*} C^*Q \right)$.

Proof. Using the van Kampen duality, we get

$$CP * CQ \simeq C^*CP \times CQ \cup_{CP \times CQ} CP \times C^*CQ \simeq C(C^*P \bar{*} Q) \cup_{C(P \bar{*} Q)} C(P \bar{*} C^*Q).$$

□

2.38. Infinite product and join. Similarly to the case of two factors one defines the *product* $\prod_{\lambda \in \Lambda} P_\lambda$ of an arbitrary family of preposets P_λ . Similarly, their *join* is defined by $\bigstar_{\lambda \in \Lambda} P_\lambda = \partial^* \prod_{\lambda \in \Lambda} C^*P_\lambda$. Note the *van Kampen duality* $\bigstar_{\lambda \in \Lambda} P_\lambda \simeq (\partial \prod_{\lambda \in \Lambda} CP_\lambda^*)^*$.

2.39. Star and link. If P is a preposet and $\sigma \in P$, we define the *star* $\text{st}(\sigma, P) = \lceil \lfloor \sigma \rfloor \rceil$ and the *link* $\text{lk}(\sigma, P) = \partial^* \lfloor \sigma \rfloor$. Thus if P is a poset, $\text{st}(\sigma, P)$ is a closed subposet of P and $\text{lk}(\sigma, P)$ is an open subposet of P .

If K is a simplicial complex, and $\sigma \in K$, then $\text{lk}(\sigma, K) = \partial^* \lfloor \sigma \rfloor$ is isomorphic to the classical link $\text{Lk}(\sigma, K) := \lceil \lfloor \sigma \rfloor \rceil \setminus \lfloor \sigma \rfloor$, which is a subcomplex of K ; an isomorphism is given by $\sigma \sqcup \tau \mapsto \tau$. It follows that $\text{st}(\sigma, K) \simeq \lceil \sigma \rceil * \text{lk}(\sigma, K)$ for every *simplicial* complex K . See [37; p. 12] for a discussion of lk versus Lk .

For an element p of a poset P , it is easy to see that $\text{lk}(p, P)^\flat = \text{Lk}(p_-, P^\flat)$ and $\text{lk}(p^*, P^*)^\flat = \text{Lk}(p_+, P^\flat)$ as subcomplexes of P^\flat (not just up to isomorphism), where p_- is a maximal chain in P with greatest element p (i.e. a maximal simplex of $\lceil p \rceil^\flat$), and p_+ is a maximal chain in P with least element p (i.e. a maximal simplex of $\lfloor p \rfloor^\flat$).

Given $\sigma \in P$ and $\tau \in Q$, clearly $\lfloor (\sigma, \tau) \rfloor^{P \times Q} \simeq \lfloor \sigma \rfloor \times \lfloor \tau \rfloor$; applying the coboundary, we obtain

$$\text{lk}((\sigma, \tau), P \times Q) \simeq \text{lk}(\sigma, P) * \text{lk}(\tau, Q).$$

Remark 2.40. In the case where P and Q are finite simplicial complexes, the latter isomorphism can be regarded as a combinatorial form of the well-known (cf. [62; 4.3.21])

homeomorphism

$$\mathrm{lk}((x, y), X \times Y) \cong \mathrm{lk}(x, X) * \mathrm{lk}(y, Y),$$

where $|P| = X$ and $|Q| = Y$ are compact polyhedra. However it does not quite fit in the familiar simplicial realm even in this case, for $P \times Q$ is no longer a simplicial complex.

2.41. Cubosimplicial complex. By a *cubosimplicial complex* we mean a conditionally complete cone complex where every cone is isomorphic to a product of simplices.

Lemma 2.42. *Let $f: K \rightarrow L$ be a simplicial map of simplicial complexes. Then $f^{-1}(\sigma)$ is a cubosimplicial complex for each $\sigma \in L$.*

Proof. By Lemma 2.34, $F := f^{-1}(\sigma)$ is full in K . Hence by Lemma 2.20, F is conditionally complete. Let $\tau \in F$, and let g be the restriction of f to the simplex $[\tau]_K$. Since $[\tau]_F = [\tau]_K \cap F = g^{-1}(\sigma)$, it suffices to show that $g^{-1}(\sigma)$ is isomorphic to a product of simplices. Viewing σ as the set of vertices of the simplex $[\sigma]$, let $\Delta_\lambda = g^{-1}(\lambda)$ for each $\lambda \in \sigma$. Then $g: [\tau] = \bigstar_{\lambda \in \sigma} \Delta_\lambda \rightarrow \bigstar_{\lambda \in \sigma} \{\lambda\} = [\sigma]$ is the join of the constant maps $g|_{\Delta_\lambda}: \Delta_\lambda \rightarrow \{\lambda\}$. Each $g|_{\Delta_\lambda}$ is the restriction of $G_\lambda: C^* \Delta_\lambda \rightarrow C^* \{\lambda\}$. Then g is the restriction of their product

$$G: C^*[\tau] = \prod_{\lambda \in \sigma} C^* \Delta_\lambda \rightarrow \prod_{\lambda \in \sigma} C^* \{\lambda\} = C^*[\sigma],$$

where the identifications come from the definition of join: $C^* \bigstar_\lambda P_\lambda = \prod_\lambda C^* P_\lambda$. Hence $g^{-1}(\sigma) = G^{-1}(\sigma) = \prod_{\lambda \in \sigma} (G_\lambda)^{-1}(\lambda) = \prod_{\lambda \in \sigma} \Delta_\lambda$. Thus $g^{-1}(\sigma)$ is a product of simplices. \square

2.43. Cubical map. A monotone map $f: P \rightarrow Q$ of posets is called *cubical* if the restriction of f to every cone of P is isomorphic to the projection $CX \times CY \rightarrow CX$ for some posets X and Y .

The following lemma ensures that a monotone map of cubosimplicial complexes is cubical if and only if its restriction to every cone is isomorphic to the projection of a product of simplices onto its subproduct.

Lemma 2.44. *If a product of simplices is isomorphic to $P \times Q$, then P and Q are products of simplices.*

Proof. We have $\mathrm{lk}(\hat{0}, P^* \times Q^*) \simeq \mathrm{lk}(\hat{0}, P^*) * \mathrm{lk}(\hat{0}, Q^*)$. On the other hand this is a join of boundaries of simplices $\partial \Delta^i$ (using that the boundary of a simplex is isomorphic to its dual), in particular a simplicial complex. If not all vertices of a $\partial \Delta^i$ are in the same factor of the join $\mathrm{lk}(\hat{0}, P^*) * \mathrm{lk}(\hat{0}, Q^*)$, then $\partial \Delta^i$ is itself a nontrivial join of simplicial subcomplexes. This is a contradiction. \square

2.45. Hatcher maps. This is a combinatorial version of Hatcher's construction [39; p. 105], [63]. Let $f: P \rightarrow Q$ be a full map of posets. If $q \in Q$, let us write $F_q = f^{-1}(q)$. Given a $p \in P$ and a $q \in Q$ such that $q < f(p)$, we have $[p]_P \cap f^{-1}(q) = [p_q]_{F_q}$ for some $p_q \in F_q$. Given a pair of elements $r, q \in Q$ such that $r > q$, define a monotone map $f_{rq}: F_r \rightarrow F_q$ by $p \mapsto p_q$.

Note that if $r > s > t$, then f_{rt} equals the composition $f^{-1}(r) \xrightarrow{f_{rs}} f^{-1}(s) \xrightarrow{f_{st}} f^{-1}(t)$.

It follows from the proof of Lemma 2.42 that

Lemma 2.46. *Hatcher maps of a simplicial map are cubical.*

We call a poset P *nonsingular* if it contains no interval of cardinality three.

Lemma 2.47. *Let $f: P \rightarrow Q$ be a full map. If P is nonsingular, then the Hatcher maps of f are closed.*

Proof. Let us show that each f_{rq} is closed. Without loss of generality r covers q . Given a $p \in F_r$ and an $s \in \partial[p_q]_{F_q}$, we have $s < p_q < p$. We need to show that $s = t_q$ for some $t \in F_q$. Arguing by induction, we may assume that p_q covers s . If p does not cover p_q , there exists a $t < p$, $t \in F_r$ (using that r covers q) such that $t_q = t_p$. By considering such a minimal t we may assume that p covers p_q . By the hypothesis there exists a $t \in P$, $t \neq p_q$, such that $s < t < p$. Then t is incomparable with p_q , whence $t \notin F_q$. Since r covers q , we conclude that $t \in F_r$. Since $p > t > s$, we have $p_q \geq t_q \geq s$. Since t is incomparable with p_q , we have $t_q \neq p_q$. Since p_q covers s , this implies that $t_q = s$. \square

2.D. Intervals and cubes

2.48. Canonical subdivision. If $P = (\mathcal{P}, \leq)$ is a [pre]poset and $a, b \in \mathcal{P}$ are such that $a \leq b$, the *interval* $[a, b]$ is the sub[pre]poset $[a] \cap [b] = \{c \in P \mid a \leq c \leq b\}$ of P . If P is a poset, we define the *canonical subdivision* $P^\#$ of P to be the poset of all intervals of P ordered by inclusion.

In the general case some additional care is needed. We say that an interval $[a, b]$ is *pre-included* in an interval $[c, d]$ and write $[a, b] \subseteq [c, d]$ if $a \in [c, d]$ and $b \in [c, d]$. When P is a poset, this is just the usual inclusion relation. In general, \subseteq is a reflexive acyclic relation on the set of all intervals of P (the latter set is really just the relation \leq which officially is a subset of $\mathcal{P} \times \mathcal{P}$). We define the *canonical subdivision* $P^\#$ of P to be the preposet of all intervals of P ordered by pre-inclusion.

We note that the canonical subdivision of every poset is an atomic poset. If P is a conditionally complete poset, then so is $P^\#$. Clearly, $(P^*)^\# \simeq P^\#$ and $(P \times Q)^\# = P^\# \times Q^\#$ for all preposets P and Q . It follows that $(C^*(P * Q))^\# = (C^*P)^\# \times (C^*Q)^\#$ and in particular, $(P * Q)^\# = (C^*P)^\# \times Q^\# \cup P^\# \times (C^*Q)^\#$.

Beware that if Q is a full subposet of P , then $Q^\#$ need not be full in $P^\#$.

Remark 2.49. In the case where P is a poset, the operation of canonical subdivision is known (under different names) in Topological Combinatorics (see Babson, Billera and Chan [9] and references there, and Živaljević [69; Definition 7]), as well as in Order Theory (see [47] and references there). Geometric versions of this construction, mostly restricted to the case where P is a simplicial complex, are also known in Algebraic Topology (see [29]), in Combinatorial Geometry (see [19]) and in Geometric Group Theory (see [12]).

2.50. Cubical complexes. The poset $I^S := (2^S)^\#$ is called a *cube* or the S -*cube*; or the n -cube (notation: I^n) if $S = [n]$. Note that

$$I^S = (C^* \Delta^S)^\# \simeq (C^* (*\Delta^0))^\# = \left(\prod_S C^* \Delta^0 \right)^\# = \prod_S (C^* \Delta^0)^\# = \prod_S I^1.$$

In particular, $I^{S \sqcup T} \simeq I^S \times I^T$ by an isomorphism natural in S and T .

A *cubical* poset is a conditionally complete poset where every cone is isomorphic to a cube. (Compare [9].) A “cubical cone complex” is abbreviated to a *cubical complex*. Cones of a cubical complex are thus called its *cubes*. Closed subposets of a cubical complex will be termed its *subcomplexes*. By the above, the product of two cubical complexes is a cubical complex. Every cubical complex is atomic, since every cube is.

In contrast to Theorem 2.28, not every cubical complex is isomorphic to a subcomplex of a cube. For instance, the simplicial complex $\partial\Delta^2$, which also happens to be a cubical complex, is not isomorphic to any subcomplex of any cube, as it contains “a cycle of odd length”.

Lemma 2.51. (a) *If K is a simplicial complex, then $(C^*K)^\#$ is a cubical complex.*

(b) *If Q is a cubical complex and $\sigma \in Q$, then $\text{lk}(\sigma, Q)$ is a simplicial complex and $\text{st}(\sigma, Q) \simeq [\sigma] \times (C^* \text{lk}(\sigma, Q))^\#$.*

Proof. (a). By Theorem 2.28, K is isomorphic to a subcomplex of some simplex Δ^S . Then $(C^*K)^\#$ is isomorphic to a subcomplex of the cube $(C^*\Delta^S)^\# \simeq I^S$. \square

(b). Given a $\tau \in \text{lk}(\sigma, Q)$, we have $([\tau], [\sigma]) \simeq (I^T, I^S)$ for some sets $S \subset T$. Then $\text{lk}(\sigma, [\tau])$ can be identified with the poset, embedded in I^T and consisting of all intervals strictly containing $[\emptyset, S]$ — that is, of all intervals $[\emptyset, S \cup R]$, where $\emptyset \neq R \subset T \setminus S$. Hence it is isomorphic to $\Delta^{T \setminus S}$.

We have $\text{st}(\sigma, [\tau]) \simeq [\sigma] \times I^{T \setminus S}$, whereas $I^{T \setminus S} \simeq (C^*\Delta^{T \setminus S})^\# \simeq (C^* \text{lk}(\sigma, [\tau]))^\#$. The resulting isomorphism $\text{st}(\sigma, [\tau]) \simeq [\sigma] \times (C^* \text{lk}(\sigma, [\tau]))^\#$ is natural in τ , which implies the second assertion.

To complete the proof of the first assertion, we note that each cone of $\text{lk}(\sigma, Q)$ is of the form $\text{lk}(\sigma, [\tau])$, which in turn has been shown to be isomorphic to a simplex. Now $\text{lk}(\sigma, Q)$ is an open subposet of the conditionally complete poset Q , and hence itself a conditionally complete poset. \square

Theorem 2.52 ([9] (see also [27], [16; 5.22], [42; Fig. 2]; compare [32])). *If K is a finite simplicial complex, there exists a finite cubical complex Q such that $\text{lk}(v, Q)$ is isomorphic to K for every vertex v of Q .*

The ‘mirroring’ construction of Theorem 2.52 has a complex analogue, where the $(\mathbb{Z}/2)^n$ symmetry is replaced by an $(S^1)^n$ symmetry; it is known as the ‘moment-angle complex’ (see [19]). Both constructions are special cases of the ‘polyhedral product’ [23].

Proof. K can be identified with a subcomplex of the $(n-1)$ -simplex $\Delta^{[n]}$ for some n . Then C^*K is identified with a closed subposet of $2^{[n]}$. The ‘folding’ monotone map $f: I \rightarrow 2^{[1]}$

can be multiplied by itself to yield a monotone map $F: I^n \rightarrow 2^{[n]}$. Let $Q = F^{-1}(C^*K)$. Then Q is a subcomplex of I^n and contains the set $F^{-1}(\hat{0})$ of all vertices of I^n . For each vertex $v = [S, S]$ in this set, define $J_v: 2^{[n]} \rightarrow I^n$ by $T \mapsto [T \setminus S, T \cup S]$. Then J_v is a monotone map, $J_v(2^{[n]}) = [v]$, and the composition $2^{[n]} \xrightarrow{J_v} I^n \xrightarrow{F} 2^{[n]}$ is the identity. Hence $F|_{[v]}$ is an isomorphism. Therefore $\text{lk}(v, Q) \simeq \text{lk}(\hat{0}, C^*K) = K$. \square

2.53. Weak product and join. If we are given some basepoints $b_\lambda \in P_\lambda$, there is the *pointed weak product* $\prod_{\lambda \in \Lambda}^w P_\lambda$, which is embedded in $\prod_{\lambda \in \Lambda} P_\lambda$ and is the union of $(\prod_{\lambda \in \Phi} P_\lambda) \times (\prod_{\lambda \in \Lambda \setminus \Phi} \{b_\lambda\})$ over all finite $\Phi \subset \Lambda$. It also has the basepoint $(b_\lambda)_{\lambda \in \Lambda}$.

The *weak join* $\bigstar_{\lambda \in \Lambda}^w P_\lambda$ is by definition $\partial^*(\prod_{\lambda \in \Lambda}^w C^*P_\lambda)$, where all the basepoints are taken at $\hat{0}$. Thus the weak join is the union of all finite subjoins. It can be identified with $\bigcup_{\lambda \in \Lambda} (P_\lambda \times \prod_{\kappa \in \Lambda \setminus \{\lambda\}}^w C^*P_\kappa)$. The *weak van Kampen duality* reads $\bigstar_{\lambda \in \Lambda}^w P_\lambda \simeq (\partial \prod_{\lambda \in \Lambda}^w C^*P_\lambda)^*$, where all basepoints are taken at $\hat{1}$.

The *weak Λ -simplex* $\Delta_w^\Lambda := \bigstar_\Lambda^w \Delta^0$ is identified with the poset $\partial^* 2_w^\Lambda$ of all nonempty finite subsets of Λ ; it has no greatest element whenever Λ is infinite. Neither has the *weak Λ -cube* $I_w^\Lambda := (2_w^\Lambda)^\# \simeq \prod_\Lambda^w I^1$, where all basepoints are taken at $[\hat{0}, \hat{0}]$. In contrast, the *co-weak Λ -cube* $I_c^\Lambda := \prod_\Lambda^w I^1$, where all basepoints are taken at $[\hat{0}, \hat{1}]$, has a greatest element. Moreover, by virtue of the weak van Kampen duality, $\partial I_c^\Lambda \simeq (S_w^\Lambda)^*$, where the *weak Λ -sphere* $S_w^\Lambda = \bigstar_\Lambda^w \partial I^1$. Somewhat reminiscent of the mirroring construction (in the proof of Theorem 2.52), S_w^Λ contains copies of Δ_w^Λ , and therefore $(I_c^\Lambda)^*$ contains copies of 2_w^Λ . In particular, $(I_c^\Lambda)^\# \simeq ((I_c^\Lambda)^*)^\#$ contains copies of $(2_w^\Lambda)^\# = I_w^\Lambda$.

2.54. Simple posets. A *simple poset* is a conditionally complete poset P such that for every $\sigma \in P$ and every $\tau \in \text{lk}(\sigma, P)$, the poset $\text{lk}(\sigma, [\tau])$ is isomorphic to a simplex. This includes simplicial and cubical posets as special cases.

Lemma 2.55. (a) If K is a simplicial poset, then C^*K is a simple poset.

(b) If P is a simple poset, $\sigma \in P$, then $\text{lk}(\sigma, P)$ is a simplicial poset.

(c) If P is a conditionally complete poset, then it is a simple poset iff $P^\#$ is a cubical poset.

(d) If P is a simple poset, then P^* is a simple poset.

(e) The face poset of a simple polytope is a simple poset.

A version of the ‘only if’ implication in (c) is found in [9].

Proof. (a). Since K is conditionally complete, then so is C^*K . If $\tau \in C^*K$, then either $\tau \in K$ or $\tau = \hat{0}$. If further $\sigma \in \text{lk}(\tau, C^*K)$, then $\text{lk}(\tau, [\sigma]_{C^*K})$ is isomorphic to the simplex $\text{lk}(\tau, [\sigma]_K)$ in the first case and to the simplex $[\sigma]$ in the second case. \square

(b). Similar to the proof of the first assertion of Lemma 2.51(b). \square

(c). Given a $\sigma \in P$, by (b), $[\sigma] \simeq C^*K$ for some simplicial poset K . Hence $[\sigma]^\#$ is a cubical poset by Lemma 2.51(a). Thus cones of $P^\#$ are cubes, and each $[\sigma]^\#$ is

conditionally complete. Now suppose that $P^\#$ contains a lower bound for a collection of intervals $[\sigma_\lambda, \tau_\lambda]$. Since P is conditionally complete, $\bigcap_\lambda \lfloor \sigma_\lambda \rfloor = \lfloor \sigma \rfloor$ for some $\sigma \in P$. Hence $\bigcap_\lambda [\sigma_\lambda, \tau_\lambda] = \bigcap [\sigma, \tau_\lambda]$, which is an interval since $\lfloor \sigma \rfloor^\#$ is conditionally complete. Thus $P^\#$ is conditionally complete.

Conversely, suppose that $P^\#$ is a cubical poset. Since P is a poset, $\lfloor \sigma \rfloor^\#$ is a subcomplex of $P^\#$ for each $\sigma \in P$, and in particular a cubical poset. Then by Lemma 2.51(b), each $\text{lk}(\lfloor \sigma, \sigma \rfloor, \lfloor \sigma \rfloor^\#)$ is a simplicial poset. On the other hand, it is isomorphic to $\text{lk}(\sigma, P)$ since P is a poset. Thus P is simple. \square

(d). Both the hypothesis and the conclusion are equivalent to saying that P is conditionally complete and every interval $[\sigma, \tau] \in P$ considered as a subposet is isomorphic to the dual cone over a simplex (\Leftrightarrow the cone over the dual of a simplex). \square

(e). If P is a convex polytope, its face poset $F_P = C(\partial F_P) = (C^*(\partial F_{P^*}))^*$, where ∂F_{P^*} is a simplicial complex if P is simple (see Example 1.12). So the assertion follows from (a) and (d). \square

2.56. Affine polytopal complexes. An *affine polytopal complex* (compare [11; last remark to Definition 2.1]) is a countable conditionally complete poset K where to every $\sigma \in K$ there is associated an isomorphism of $\lfloor \sigma \rfloor$ with the poset of non-empty faces of a convex polytope P_σ (compare Example 1.11) so that every face inclusion $\lfloor \tau \rfloor \subset \lfloor \sigma \rfloor$ is realized by an affine isomorphism between P_τ and the face of P_σ corresponding to τ . This includes cubical and simplicial complexes as special cases. Every affine polytopal complex is atomic since the poset of nonempty faces of every convex polytope is. Affine polytopal complexes are not equivalent to “cell complexes” in the sense of Rourke–Sanderson [57], who additionally require linear embeddability into some Euclidean space. For instance, the cubulation of the Möbius band into 3 squares is an affine polytopal complex (and a cubical complex) but not a cell complex in the sense of [57] (see [19]).

We note the following consequence of Lemma 2.55(c,e), which can be seen as a combinatorial abstraction of a well-known geometric construction (see [19]).

Corollary 2.57. *If K is an affine polytopal complex whose polytopes are simple, then K is a simple poset. In particular, $K^\#$ is a cubical complex.*

2.E. Mapping cylinder

2.58. Adjunction preposet. Let $P = (\mathcal{P}, \leq)$ be a poset, $A = (\mathcal{A}, \leq)$ a subposet of P and $f: A \rightarrow Q$ a monotone map, where $Q = (\mathcal{Q}, \leq)$ is a preposet. We recall that the adjunction set $\mathcal{P} \cup_f \mathcal{Q}$ is the set-theoretic pushout of the diagram $\mathcal{P} \supset \mathcal{A} \xrightarrow{f} \mathcal{Q}$, that is, the quotient of $\mathcal{P} \sqcup \mathcal{Q}$ by the equivalence relation generated by $a \sim f(a)$ for all $a \in \mathcal{A}$.

We define the *adjunction preposet* $P \cup_f Q$ to be the preposet $P \cup_f Q = (\mathcal{P} \cup_f \mathcal{Q}, \leq)$, where $q \leq q'$ in $P \cup_f Q$ if there exist $p \in [q]$ and $p' \in [q']$ such that $p \leq p'$. It is easy to see that this relation is reflexive and acyclic (given a nontrivial cycle in $P \cup_f Q$, it can

be shortened to a cycle in Q using that P is a poset). It is also clear that $P \cup_f Q$ is the pushout of the diagram $P \supset A \xrightarrow{f} Q$ in the category of preposets.

Lemma 2.59. *Let $P \supset A \xrightarrow{f} Q$ be a partial monotone map of posets.*

- (a) *If f is closed and A is closed, then $P \cup_f Q$ is a poset.*
- (b) *If f is open and A is open, then $P \cup_f Q$ is a poset.*

Proof. Since P and Q are posets, it suffices to check transitivity for triples of the form $[p] > [a] = [f(a)] > [q]$ and $[p] < [a] = [f(a)] < [q]$, where $p \in P$, $a \in A$ and $q \in Q$.

Suppose that $p \geq a$ and $f(a) \geq q$. If f is closed, then $q = f(b)$ for some $b \leq a$. Then $p \geq b$, and consequently $[p] \geq [b] = [q]$ in $P \cup_f Q$. On the other hand, if A is open, then $p \in A$, and consequently $f(p) \geq f(a) \geq q$. Since Q is a poset, $f(p) \geq q$, and thus $[p] = [f(p)] \geq [q]$.

Now suppose that $p \leq a$ and $f(a) \leq q$. If f is open, then $q = f(b)$ for some $b \geq a$. Then $p \geq b$, and consequently $[p] \leq [b] = [q]$ in $P \cup_f Q$. On the other hand, if A is closed, then $p \in A$, and consequently $f(p) \leq f(a) \leq q$. Since Q is a poset, $f(p) \leq q$, and thus $[p] = [f(p)] \leq [q]$. \square

2.60. Quotient poset. Given a subposet Q of a poset P , it is easy to see that the adjunction preposet $P \cup_c pt$, where $f: Q \rightarrow pt$ is the constant map, is a poset. We call it the *quotient poset* and denote P/Q .

2.61. Amalgam. If A and B are subposets of posets P and Q , respectively, and $h: A \rightarrow B$ an isomorphism, then the *amalgam* $P \cup_h Q$, also written $P \cup_{A=B} Q$ when h is clear from context or irrelevant, is the adjunction preposet $P \cup_{jh} Q$, where $j: B \rightarrow Q$ is the inclusion.

Corollary 2.62. *Let A and B be subposets of posets P and Q , respectively. If A and B are both closed or both open, then $P \cup_{A=B} Q$ is a poset.*

2.63. Mapping cylinder. Given a monotone map of posets $f: P \rightarrow Q$, the *mapping cylinder* $MC(f) = P \times [2] \cup_{f_1} Q$ and the *dual mapping cylinder* $MC^*(f) = P \times [2] \cup_{f_2} Q$, where f_i is the composition $P \times \{i\} \simeq P \xrightarrow{f} Q$. Note that $MC^*(f) \simeq (MC(f^*))^*$ and that there are natural monotone bijections (which are not embeddings) $MC(f) \rightarrow Q + P$ and $MC^*(f) \rightarrow P + Q$.

Note that

$$P * Q \simeq MC(P \times Q \rightarrow P) \bigcup_{P \times Q} MC(P \times Q \rightarrow Q)$$

and

$$P \bar{*} Q \simeq MC^*(P \times Q \rightarrow P) \bigcup_{P \times Q} MC^*(P \times Q \rightarrow Q).$$

Corollary 2.64. *Let $f: P \rightarrow Q$ be a monotone map of posets. Then $MC(f)$ is a poset if and only if f is closed; dually, $MC^*(f)$ is a poset if and only if f is open.*

Proof. If f is closed, then $MC(f)$ is a poset by Lemma 2.59. If f is not closed, there exist a $p \in P$ and a $q < f(p)$ such that $q \neq f(p')$ for any $p' < p$. Then $(p, 2) > (p, 1) = f(p) > q$ but $(p, 2) \not\geq q$ in $MC(f)$. The dual assertion follows from $MC^*(f) \simeq (MC(f^*))^*$. \square

Lemma 2.65. *Let $\psi: Q' \rightarrow Q$ be a closed map of posets. Consider pullback diagrams*

$$\begin{array}{ccc} P' & \xrightarrow{\varphi} & P \\ f' \downarrow & & f \downarrow \\ Q' & \xrightarrow{\psi} & Q \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\Phi} & P \\ F \downarrow & & f \downarrow \\ MC(\psi) & \xrightarrow{\psi \cup \text{id}_Q} & Q. \end{array}$$

Then

- (a) φ is closed;
- (b) $X \simeq MC(\varphi)$, $\Phi \simeq \varphi \cup \text{id}_P$ and $F \simeq f' \cup f$.

Proof. (a). Let $(p, q') \in P' \subset P \times Q'$. Then $f(p) = \psi(q')$, and $\varphi((p, q')) = p$. Let $p_0 \leq p$. Then $f(p_0) \leq \psi(q')$. Since ψ is closed there exists a $q'_0 \leq q'$ such that $\psi(q'_0) = f(p_0)$. Thus $(p_0, q'_0) \in P'$ and $(p_0, q'_0) \leq (p, q')$. \square

(b). Clearly, X is a union of an open copy of P' and a closed copy of P . It suffices to check that the resulting bijection between X and $MC(\varphi)$ is monotone. Let $(p', q') \in P' \subset P \times Q'$, so that $f(p') = \psi(q')$, and let $(p, f(p)) \in \Gamma_f \subset P \times Q$, where Γ_f is the graph of f . Then $(p', q') > (p, f(p))$ in $P \times MC(\psi)$ if and only if $p' \geq p$ in P and $q' > f(p)$ in $MC(\psi)$. We recall that $q' > f(p)$ in $MC(\psi)$ if and only if $\psi(q') \geq f(p)$. But the latter follows from $p' \geq p$ due to our assumption $\psi(q') = f(p')$. On the other hand, $(p', q') > p$ in $MC(\varphi)$ if and only if $\varphi((p', q')) \geq p$. But $\varphi((p', q')) = p'$, and the assertion follows. \square

2.66. Long mapping cylinder. Given a monotone map $f: P \rightarrow Q$ between preposets, let $LMC(f) = MC(f) \cup_{P=P \times \{1\}} P \times [2]$.

Lemma 2.67. *Let $f: P \rightarrow Q$ be a monotone map between conditionally complete posets that preserves infima. Then*

- (a) $\langle MC(f) \rangle$ is conditionally complete.
- (b) $\langle LMC(f) \rangle$ is conditionally complete.

Note that $MC(pt \sqcup pt \rightarrow pt) \cup_{pt \sqcup pt} MC(pt \sqcup pt \rightarrow pt)$ is not conditionally complete.

Dually to (a), if f preserves suprema, then $\langle MC^*(f) \rangle$ is conditionally complete (using that $MC^*(f) = (MC(f^*))^*$).

Proof. (a). Let S be a nonempty subset of $\langle MC(f) \rangle$. Suppose that $q \in Q$ and $s \in S \cap P$. If $q \leq f(s)$, then $q \leq s$. Conversely, if $q \leq s$, then $q \leq f(p)$ for some $p \leq s$; hence $q \leq f(s)$. Thus lower bounds of S that lie in Q coincide with lower bounds of $(S \cap Q) \cup f(S \cap P)$.

If every lower bound of S belongs to Q , then the set of lower bounds of S coincides with that of $(S \cap Q) \cup f(S \cap P)$. If this set is nonempty, then it has a greatest element (since Q is conditionally complete).

It remains to consider the case where S has a lower bound in P . Then S has a greatest lower bound p in P (since P is conditionally complete). Since f preserves infima, $f(p)$ is a greatest lower bound of $f(S)$. If $q \in Q$ is a lower bound of S , then it is a lower bound of $f(S)$, and hence $q \leq f(p) \leq p$. Thus p is globally a greatest lower bound of S . \square

(b). Let S be a nonempty subset of $\langle LMC(f) \rangle$. If S has a lower bound $(p, 0)$ in $P \times \{0\}$ and a lower bound in Q , then $S \subset P \times \{1\}$, and consequently $(p, 1)$ is also a lower bound of S . Hence either every lower bound of S belongs to Q , or every lower bound of S belongs to R , or S has a lower bound in P . Each of the three cases is considered similarly to the proof of (a). \square

Example 2.68. Let $f: \Delta^2 \rightarrow \Delta^1$ be a simplicial surjection. It is easy to see that f preserves suprema. Being simplicial, f is closed, so $MC(f)$ is a poset; but not a conditionally complete one.

Indeed, writing f as $\{a\} * \{b, c\} \rightarrow \{a\} * \{d\}$, we see that the edges $\{a, b\}$ and $\{a, c\}$ in the domain have a greatest lower bound in the domain (namely, their common vertex $\{a\}$) and a greatest element among lower bounds in the range (namely, their common image $\{a, d\}$), but no greatest lower bound in the entire mapping cylinder.

The problem persists for the mapping cylinder $MC(f^{\#n})$ of the iterated canonical subdivision $f^{\#n}: (\Delta^2)^{\#n} \rightarrow (\Delta^1)^{\#n}$.

2.69. Thick mapping cylinder. Let $f: P \rightarrow Q$ be a monotone map of posets. Let us consider $R := \lceil \Gamma(f) \rceil_{P \times Q}$, the closure of the graph $\Gamma(f) = \{(p, f(p)) \mid p \in P\}$ in $P \times Q$. Let π_P and π_Q be the projections onto the factors of $P \times Q$. Being cubical, they are closed maps, and since R is closed in $P \times Q$, so are their restrictions to R . Hence $TMC(f) = MC(\pi_P|_R) \cup_R MC(\pi_Q|_R)$ is a poset. It follows then that $TMC(f)$ is a closed subposet of $P * Q$. From this we get that if P and Q are conditionally complete, then so is $TMC(f)$.

Theorem 2.70. *If $f: P \rightarrow Q$ is a monotone map between posets, where Q is conditionally complete, then $TMC(f)$ strong deformation retracts onto $LMC(f)$ via a monotone map $r: TMC(f) \rightarrow LMC(f)$, homotopic to the identity by a monotone homotopy $H: TMC(f) \times I \rightarrow TMC(f)$ that extends the projection $LMC(f) \times I \rightarrow LMC(f)$.*

Proof. Let $R = \lceil \Gamma(f) \rceil_{P \times Q}$. Given a $(p, q) \in R$, we have $(p, q) \leq (x_{pq}, f(x_{pq}))$ for some $x_{pq} \in P$. Then $q \leq f(x_{pq})$ and $f(p) \leq f(x_{pq})$, so $f(x_{pq})$ is an upper bound for q and $f(p)$. Let y_{pq} be the least upper bound for q and $f(p)$. Let us define $g: R \rightarrow R$ by $(p, q) \mapsto (p, y_{pq})$. If $(p, q) \leq (p', q')$, then $q \leq q'$ and $f(p) \leq f(p')$, so $y_{p'q'}$ is an upper bound for q and $f(p)$. Hence $y_{pq} \leq y_{p'q'}$, and thus g is monotone.

Let r be the composition $P \times Q \xrightarrow{\pi_P|} P \xrightarrow{\text{id}_P \times f} \Gamma(f)$. Since $g((p, q)) \geq (p, q)$ and $g((p, q)) \geq (p, f(p)) = r((p, q))$, the maps id_R , g and $r|_R$ combine into a monotone map $h: R \times I \rightarrow R$. Since $\pi_P(g(p, q)) \geq p$ and $\pi_Q(g(p, q)) \geq q$, the map h and the projection $(P \sqcup Q) \times I \rightarrow P \sqcup Q$ combine into a monotone map $H: TMC(f) \times I \rightarrow TMC(f)$. \square

2.71. Iterated mapping cylinder. For a monotone map $f: P \rightarrow Q$, let $\pi: MC(f) \rightarrow Q$ and $\pi: MC^*(f) \rightarrow Q$ be defined by $\pi|_P = f$ and $\pi|_Q = \text{id}$. If f is closed [resp. open], clearly $\pi: MC(f) \rightarrow Q$ is closed [resp. $\pi: MC^*(f) \rightarrow Q$ is open]. Let us define the preposet $MC(P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0) = MC(F_n)$, where F_n is the composition $MC(P_n \xrightarrow{f_n} \dots \xrightarrow{f_2} P_1) \xrightarrow{\pi} P_1 \xrightarrow{f_1} P_0$. Similarly $MC^*(P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0) = MC^*(F_n)$, where F_n is the

composition $MC^*(P_n \xrightarrow{f_n} \dots \xrightarrow{f_2} P_1) \xrightarrow{\pi} P_1 \xrightarrow{f_1} P_0$. If f_1, \dots, f_n are closed [resp. open], clearly $MC(P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0)$ [resp. $MC^*(P_n \xrightarrow{f_n} \dots \xrightarrow{f_1} P_0)$] is a poset.

2.72. Homotopy colimit. Suppose we have a covariant [resp. contravariant] commutative diagram, indexed by a poset Λ , consisting of posets $P_\lambda = (\mathcal{P}_\lambda, \leq)$, $\lambda \in \Lambda$, and monotone maps $f_{\lambda\mu}: P_\lambda \rightarrow P_\mu$ whenever $\lambda > \mu$ [resp. $\lambda < \mu$] in Λ . Then the *homotopy colimit* $\text{hocolim}_\Lambda(P_\lambda, f_{\lambda\mu})$ is the preposet $Q = (\mathcal{Q}, \leq)$, where $\mathcal{Q} = \bigsqcup_{\lambda \in \Lambda} \mathcal{P}_\lambda$, and given a $p \in \mathcal{P}_\lambda$ and a $q \in \mathcal{P}_\mu$, we have $p \geq q$ in Q if and only if either $\lambda = \mu$ and $p \geq q$ in P_λ , or $\lambda > \mu$ and there exists a $p' \in \mathcal{P}_\lambda$ such that $p' \leq p$ in P_λ and $f_{\lambda\mu}(p') = q$ [resp. there exists a $q' \in \mathcal{P}_\mu$ such that $q' \geq q$ in P_μ and $f_{\mu\lambda}(q') = p$]. If the maps $f_{\lambda\mu}$ are closed [resp. open], it is clear that $\text{hocolim}_\Lambda(P_\lambda, f_{\lambda\mu})$ is a poset. Clearly, $MC(P_n \xrightarrow{f_n} \dots \xrightarrow{f_2} P_1) = \text{hocolim}_{[n]}(P_i, f_{ij})$, where $f_{ij}: P_i \rightarrow P_j$ is the composition $P_i \xrightarrow{f_i} \dots \xrightarrow{f_{j+1}} P_j$. Similarly $MC^*(P_n \xrightarrow{f_n} \dots \xrightarrow{f_2} P_1) = \text{hocolim}_{[n]}^*(P_i, f_{ij})$. In general, it is easy to see that every homotopy colimit indexed by a poset is in fact an iterated amalgam of the iterated mapping cylinders. Since every $f_{\lambda\mu}$ is a single-valued (not multi-valued) map, every subposet P_λ of $\text{hocolim}_\Lambda(P_\lambda, f_{\lambda\mu})$ is full in the homotopy colimit.

The following is straightforward from Lemma 2.47.

Theorem 2.73. *If $f: P \rightarrow Q$ is a full map and P is nonsingular (e.g. f is simplicial), then $P = \text{hocolim}_Q(f^{-1}(q), f_{rq})$, where $f_{rq}: f^{-1}(r) \rightarrow f^{-1}(q)$ are the Hatcher maps.*

The following is a combinatorial form of Homma's factorization lemma (see [59; proof of Lemma 5.4.1] and [22; Lemma 2.1]).

Corollary 2.74. *Let $f: P \rightarrow Q$ be a full map, where P is nonsingular. Then f factors as a composition of full maps f_i such that each f_i has only one non-degenerate point inverse.*

Proof. Let us extend the partial order on Q to a total order. Let $F_q^{(i)} = f^{-1}(q)$ if q is not among the first i elements of Q , and otherwise let $F_q^{(i)}$ be the singleton poset. Let $f_{rq}^{(i)}: F_r^{(i)} \rightarrow F_q^{(i)}$ be the Hatcher map if $F_q^{(i)} = f^{-1}(q)$ and the unique map if $F_q^{(i)}$ is a singleton poset. Let $P_i = \text{hocolim}_Q(F_q^{(i)}, f_{rq}^{(i)})$. Then the obvious maps $P = P_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} P_n$ are as desired. \square

The following lemma is straightforward.

Lemma 2.75. (a) *Given a covariant commutative diagram of posets P_λ and monotone maps $f_{\lambda\mu}: P_\lambda \rightarrow P_\mu$ indexed by a cone $C\Lambda$, we have*

$$\text{hocolim}_{C\Lambda}(P_\lambda, f_{\lambda\mu}) \simeq MC\left(P_1 \xleftarrow{\pi} P_1 \times \Lambda \xrightarrow{F} \text{hocolim}_\Lambda(P_\lambda, f_{\lambda\mu})\right),$$

where $F(p, \lambda) = f_{1\lambda}(p)$.

(b) *Given a contravariant commutative diagram of posets P_λ and monotone maps $f_{\lambda\mu}: P_\lambda \rightarrow P_\mu$ indexed by a dual cone $C^*\Lambda$, we have $\text{hocolim}_{C^*\Lambda}(P_\lambda, f_{\lambda\mu}) \simeq MC(F)$, where $F: \text{hocolim}_\Lambda(P_\lambda, f_{\lambda\mu}) \rightarrow P_0$ is given by $F|_{P_\lambda} = f_{\lambda 0}$.*

2.F. Canonical and barycentric handles

The *barycentric subdivision map* $\flat: P^\flat \rightarrow P$ is defined by sending every nonempty chain $\{p_1 < \dots < p_n\}$ to its maximal element p_n . The *canonical subdivision map* $\# : P^\# \rightarrow P$ is defined by sending every interval $[p, q]$ to its maximal element q . (These are instances of a general notion of subdivision map to be studied in §3.A below.)

Lemma 2.76. *Let P be a conditionally complete poset. Then $\# : P^\# \rightarrow P$ preserves infima and suprema.*

Proof. Suppose that $S \subset P^\#$ has a supremum $[p_0, q_0]$. Thus each $[p, q] \in S$ is included in $[p_0, q_0]$, and each upper bound $[p, q]$ of S includes $[p_0, q_0]$. Given an upper bound q_1 of $\#(S)$ and a $[p, q] \in S$, we have $[p, q] \subset [p_0, q_0]$ and $q \leq q_1$, hence $[p, q] \subset [p_0, q_1]$. Then $[p_0, q_1]$ is an upper bound of S , so $[p_0, q_0] \subset [p_0, q_1]$. Hence $q_0 \leq q_1$, which shows that q_0 is the supremum of $\#(S)$.

Suppose that $S \subset P^\#$ has an infimum $[p_0, q_0]$. Thus each $[p, q] \in S$ includes $[p_0, q_0]$, and each lower bound $[p, q]$ of S is included in $[p_0, q_0]$. Given a lower bound q_1 of $\#(S)$ and a $[p, q] \in S$, we have $[p_0, q_0] \subset [p, q]$ and $q_1 \leq q$, hence $[p_0, q_1] \subset [p, q]$. Then $[p_0, q_1]$ is a lower bound of S , so $[p_0, q_1] \subset [p_0, q_0]$. Hence $q_1 \leq q_0$, which shows that q_0 is the infimum of $\#(S)$. \square

Corollary 2.77. *If P is a conditionally complete poset, then $MC^*(\# : P^\# \rightarrow P)$ is a conditionally complete poset.*

2.78. Canonical handles. Let X be a poset. Then $[\sigma, \tau] < [\rho, \nu]$ in $X^\#$ iff $\rho \leq \sigma$ and $\tau \leq \nu$. The dual cone $\llbracket [\sigma, \tau] \rrbracket$ is the poset of all such intervals $[\rho, \nu]$; clearly, it is isomorphic to $(\lceil \sigma \rceil)^* \times \lfloor \tau \rfloor$. Then every cone $\lceil [\sigma, \tau]^* \rceil = (\llbracket [\sigma, \tau] \rrbracket)^*$ of the dual poset $h(X) := (X^\#)^*$ is isomorphic to the product $\lceil \sigma \rceil \times (\lfloor \tau \rfloor)^* = \lceil \sigma \rceil \times \lceil \tau^* \rceil$.

The maximal cones of $h(X)$ (i.e. those cones that are not properly contained in other cones) are of the form $\lceil [\sigma, \sigma]^* \rceil$, where $\sigma \in X$, and are called the (*canonical*) *handles* h_σ of the poset X . Each h_σ is isomorphic to the product of the cone $\lceil \sigma \rceil$ of X and the cone $\lceil \sigma^* \rceil$ of X^* . These cones are called the *core* and *cocore* of the handle h_σ .

When $\sigma < \tau$, the intersection $h_\sigma \cap h_\tau$ is clearly the cone $\lceil [\sigma, \tau]^* \rceil$ of $h(X)$. However, when σ and τ are incomparable in X , it may be that $h_\sigma \cap h_\tau$ is nonempty (namely, it is nonempty iff both $\lceil \sigma \rceil \cap \lceil \tau \rceil$ and $\lfloor \sigma \rfloor \cap \lfloor \tau \rfloor$ are nonempty). This distinguishes the canonical handles from the barycentric ones, which will be discussed in a moment.

2.79. Collapsing handles onto cores. Let X be a poset. Let us consider the composition $X^\# \xrightarrow{j_X} (X^*)^\# \xrightarrow{\#} X^*$, where the isomorphism $j_X : X^\# \rightarrow (X^*)^\#$ is given by $[\sigma, \tau] \mapsto [\tau^*, \sigma^*]$. Then the dual map $r_X : h(X) \rightarrow X$ to this composition is given by $[\sigma, \tau]^* \mapsto \sigma$. On the other hand, the composition $\bar{r}_X : h(X) \xrightarrow{(j_X)^*} h(X^*) \xrightarrow{r_{X^*}} X^*$ is dual to $\# : X^\# \rightarrow X$ and is given by $[\sigma, \tau]^* \mapsto \tau^*$.

The restrictions of r_X and \bar{r}_X to the cone $\lceil [\sigma, \tau]^* \rceil \simeq \lceil \sigma \rceil \times \lceil \tau^* \rceil$ are the projection onto the first and second factor, respectively. We also note that $h_\sigma = r_X^{-1}(\lceil \sigma \rceil)$ and $h_\tau = \bar{r}_X^{-1}(\lceil \tau^* \rceil)$.

Lemma 2.80. *Let P and Q be posets. Then*

- (a) $(P^* + Q)^\# \simeq MC(P \times Q \xleftarrow{\# \times \text{id}} P^\# \times Q \xrightarrow{\pi} P^\#) \bigcup_{P \times Q} MC(P \times Q \xleftarrow{\text{id} \times \#} P \times Q^\# \xrightarrow{\pi} Q^\#),$
 (b) $h(CP) \simeq MC(r_P) \cup_P CP.$

Proof. (a). An interval of $P^* + Q$ is either an interval of P^* (or equivalently of P) or an interval of Q , or a dual cone of P^* plus a cone of Q (or equivalently, a cone $P \times Q$). \square

(b). Part (a) specializes to $(CP)^\# \simeq MC^*((P^*)^\# \xrightarrow{\#} P^*) \cup_{P^*} MC(P^* \rightarrow pt)$. Note that $MC(P^* \rightarrow pt) \simeq C^*(P^*)$. Hence $h(CP) \simeq MC(h(P) \xrightarrow{r_P} P) \cup_P CP$. \square

2.81. Barycentric handles. Associated to every poset P is the barycentric handle decomposition $H(P) := (P^b)^*$.

Since $(P + Q)^b \simeq P^b * Q^b$, we have $H(P + Q) \simeq H(P) \bar{*} H(Q)$. In particular, $H(CP) \simeq H(P) \times [2] \cup_{H(P)} CH(P)$.

A barycentric handle corresponding to a $\sigma \in P$ is the cone $H_\sigma := \lceil(\hat{\sigma})^*\rceil$ of the maximal element $(\hat{\sigma})^*$ of $H(P)$. We write $H_\sigma = H_\sigma^P$ when P is not clear from the context.

Lemma 2.82. H_σ is isomorphic to the product of cones $CH(\partial\lceil\sigma\rceil) \times CH(\partial\lceil\sigma^*\rceil)$.

It follows, in particular, that $H_\sigma \simeq H_\sigma^{\lceil\sigma\rceil} \times H_\sigma^{\lfloor\sigma^*\rceil}$.

Proof. $\partial H_\sigma = \partial\lceil(\hat{\sigma})^*\rceil$ can be identified with $H(\partial\lceil\sigma\rceil + \partial^*\lfloor\sigma^*\rceil)$. By the above the latter is isomorphic to $H(\partial\lceil\sigma\rceil) \bar{*} H(\partial^*\lfloor\sigma^*\rceil)$. Since $H(Q) \simeq H(Q^*)$, we have $H(\partial^*\lfloor\sigma^*\rceil) \simeq H(\partial\lceil\sigma^*\rceil)$. Thus H_σ is isomorphic to the cone over $H(\partial\lceil\sigma\rceil) \bar{*} H(\partial\lceil\sigma^*\rceil)$. By the van Kampen duality, this cone is isomorphic to $CH(\partial\lceil\sigma\rceil) \times CH(\partial\lceil\sigma^*\rceil)$. \square

3. SUBDIVISION AND COLLARS

Starting with this chapter, all posets and preposets are assumed to be finite.

If P is a poset or preposet, $|P|$ will denote the polyhedron triangulated by the simplicial complex P^b . If $f: P \rightarrow Q$ is a monotone map, $|f|$ will denote the PL map triangulated by the simplicial map $f^b: P^b \rightarrow Q^b$.

To be more specific, we can canonically embed P^b into the simplex $\Delta^{\mathcal{P}}$, where \mathcal{P} is the underlying set of P (Lemma 2.25) and consider the corresponding affine simplicial subcomplex of an affine simplex whose face poset is isomorphic to $\Delta^{\mathcal{P}}$.

We will also assume all polyhedra to be compact (and endowed with PL structure apart from topology), and all maps between polyhedra to be PL.

3.A. Subdivision maps and transversality

Lemma 3.1. (a) If P is a closed suposet of a poset Q , then $|P|$ is homeomorphic to $|CQ|$ keeping $|Q|$ fixed if and only if $(|P|, |Q|)$ is homeomorphic to $(|CQ|, |Q|)$.

(b) If P is a poset, and $|P|$ is homeomorphic to $|C(\partial P)|$, then they are homeomorphic keeping $|\partial P|$ fixed.

Proof. (a). This follows from the fact that every self-homeomorphism of $|Q|$ extends to a self-homeomorphism of the cone $C|Q|$ (which in turn is homeomorphic to $|CQ|$ keeping $|Q|$ fixed). \square

(b). We recall that ∂P is the closure of the subposet of all $p \in P$ such that $\text{lk}(p, P)$ is a singleton. By considering barycentric subdivisions we may assume that P is a simplicial complex. Then by the combinatorial invariance of link any homeomorphism $|P| \rightarrow |\partial P * pt|$ sends $|\partial P|$ into itself. Thus $(|P|, |\partial P|)$ is homeomorphic to $(|C(\partial P)|, |\partial P|)$, and the assertion follows from (a). \square

3.2. Subdivision map. By a *subdivision map* we mean a monotone map $f: P' \rightarrow P$ between posets such that $|f^{-1}(\lceil p \rceil)|$ is homeomorphic to $|Cf^{-1}(\partial \lceil p \rceil)|$ by a homeomorphism fixed on $|f^{-1}(\partial \lceil p \rceil)|$ for each $p \in P$.

Clearly, every subdivision map is an open map. If $f: P' \rightarrow P$ is a subdivision map, then by the combinatorial invariance of link $\partial(P') = f^{-1}(\partial P)$.

3.3. Realizable subdivision map. If P and P' are affine polytopal [resp. simplicial] complexes, we say that a subdivision map $\alpha: P' \rightarrow P$ is *affinely* [resp. *simplicially*] *realizable* if there exists an affine subdivision $K' \triangleright K$ of affine polytopal [simplicial] complexes K and K' such that P and P' are the posets of nonempty faces of K and K' , and the smallest polytope [simplex] of K containing the polytope [simplex] of K' corresponding to a $p \in P'$ corresponds to $\alpha(p) \in P$.

Theorem 3.4. *A monotone map of posets $f: P' \rightarrow P$ is a subdivision map if and only if there exists a homeomorphism $h: |P'| \rightarrow |P|$ such that $h^{-1}(|\lceil p \rceil|) = |f^{-1}(\lceil p \rceil)|$ for each $p \in P$.*

Such a homeomorphism h will be called an *underlying homeomorphism* of the subdivision map f .

The proof of Theorem 3.4 is rather straightforward.

Proof. Assume inductively that for some closed subposet Q of P there exists an underlying homeomorphism h_Q of $f|_{Q'}$, where $Q' = f^{-1}(Q)$. Pick some $p \in P \setminus Q$ such that $R := \partial \lceil p \rceil \subset Q$. Let $Q_+ = Q \cup CR \subset P$ and $(Q_+)' = f^{-1}(Q_+)$. Let $R' = f^{-1}(R)$ and let $(Q')_+$ be the amalgam $Q' \cup_{R'} CR'$. Then $f|_{(Q_+)'}$ factors uniquely as $(Q_+)' \xrightarrow{g} (Q')_+ \xrightarrow{f_+} Q_+$, where g is a subdivision map extending $\text{id}_{Q'}$ and f_+ is a subdivision map extending $f|_{Q'}$. Then $h_{Q'}$ extends conically to an underlying homeomorphism h_+ of f_+ , and $\text{id}_{|Q'|}$ extends, using the homeomorphism from the definition of a subdivision map, to an underlying homeomorphism h_g of g . The composition $|(Q_+)'| \xrightarrow{h_g} |(Q')_+| \xrightarrow{h_+} |Q_+|$ is clearly an underlying homeomorphism of $f|_{(Q_+)'}$, which completes the induction step.

Conversely, suppose that h is an underlying homeomorphism of f . Given a $p \in P$, we have $|\partial \lceil p \rceil| = \bigcup_{q < p} |\lceil q \rceil|$, and consequently

$$h^{-1}(|\partial \lceil p \rceil|) = \bigcup_{q < p} h^{-1}(|\lceil q \rceil|) = \bigcup_{q < p} |f^{-1}(\lceil q \rceil)| = |f^{-1}(\partial \lceil p \rceil)|.$$

Hence we get the composite homeomorphism

$$|f^{-1}([p])| \xrightarrow{h|_{h^{-1}(|[p]|)}} |[p]| = C|\partial[p]| \xrightarrow{C(h^{-1}(|\partial[p]|))} C(f^{-1}(|\partial[p]|)),$$

which is the identity on $f^{-1}(|\partial[p]|)$, as desired. \square

Corollary 3.5. *Let $P'' \xrightarrow{f} P' \xrightarrow{g} P$ be monotone maps of posets, where f is a subdivision map. Then g is a subdivision map if and only if gf is.*

Remark 3.6. Subdivision maps of cell complexes were studied by Mnëv, who calls them assembly or aggregation maps [51].

Subdivisions that are *dual* to simplicial maps of simplicial complexes were studied by Akin [6], who called them transversely cellular maps, and later rediscovered by Dragotti and Magro (see [26]) under the name of strong cone-dual maps; both approaches extend Cohen's work on the case where the domain is a manifold [20].

Akin gave a nontrivial proof of Corollary 3.5 in the case of maps dual to simplicial maps [6; Corollary 3 on p. 423]. He also proved, by providing some invariant characterizations, that the property of being a subdivision map for the dual to a simplicial map does not depend on the choice of triangulations [6].

Subdivisions of posets as defined by Theorem 3.4 are easily seen to be equivalent to McCrory's notion of subdivision of a cone complex [48].

The following example is found in [6]:

Example 3.7. If g and gf are subdivision maps, f need not be one: let M be a manifold and $f = gf: M \rightarrow M$ shrink a codimension zero ball B to a point; then g must be the identity outside B but can behave arbitrarily inside B .

3.8. Fiberwise subdivision map. A subdivision map $f: P' \rightarrow P$ is called *fiberwise* with respect to monotone maps $\pi: P \rightarrow B$ and $\pi': P' \rightarrow B$ if f sends $(\pi')^{-1}([b])$ onto $\pi^{-1}([b])$ for each $b \in B$, and for each $p \in P$ there exists a homeomorphism $h_p: |f^{-1}([p])| \rightarrow C|f^{-1}(\partial[p])|$ keeping $|f^{-1}(\partial[p])|$ fixed and sending $|(\pi')^{-1}([b]) \cap f^{-1}([p])|$ onto $C|(\pi')^{-1}([b]) \cap f^{-1}(\partial[p])|$ for every $b \in B$.

In the case that π' equals the composition $P' \xrightarrow{f} P \xrightarrow{\pi} B$, we also say that the subdivision map f is *fiberwise* with respect to π , or just lies *over* B when f is clear from context.

It is easy to see that every subdivision map $f: P' \rightarrow P$ lies over the constant map $P \rightarrow pt$. On the other hand, if f lies over $\text{id}: P \rightarrow P$, then f itself is the identity map.

In dealing with transversality we will also need a peculiar variety of a fiberwise subdivision map. To define when the subdivision map f is *fiberwise* with respect to monotone maps $\pi: P \rightarrow B$ and $\pi': P' \rightarrow B^*$, we repeat the same conditions as above, but with $(\pi')^{-1}([b])$ replaced by $(\pi')^{-1}([b^*])$ throughout.

Lemma 3.9. *Let $f: P' \rightarrow P$ be a fiberwise subdivision map with respect to $\pi: P \rightarrow B$ and $\pi': P' \rightarrow B$ [resp. $\pi': P' \rightarrow B^*$], and let $B_0 \subset B$ be an open subposet. Let $P_0 = \pi^{-1}(B_0)$ and $P'_0 = (\pi')^{-1}(B_0)$ [resp. $P'_0 = (\pi')^{-1}(B_0^*)$]. Then the restriction $f_0: P'_0 \rightarrow P_0$*

of f is a subdivision map with respect to the restrictions $\pi_0: P_0 \rightarrow B_0$ and $\pi'_0: P'_0 \rightarrow B_0$ [resp. $\pi'_0: P'_0 \rightarrow B_0^*$] of π and π' . In particular, f_0 is a subdivision map.

Proof. We only treat one case; the case in square brackets is similar. Given a $p \in P_0$, the hypothesis provides a homeomorphism $h_p: |f^{-1}(\lceil p \rceil)| \rightarrow C|f^{-1}(\partial \lceil p \rceil)|$ keeping $|f^{-1}(\partial \lceil p \rceil)|$ fixed and sending $|P'_0 \cap f^{-1}(\lceil p \rceil)|$ onto $C|P'_0 \cap f^{-1}(\partial \lceil p \rceil)|$. We have $P'_0 \cap f^{-1}(\lceil p \rceil) = f_0^{-1}(P_0 \cap \lceil p \rceil) = f_0^{-1}(\lceil p \rceil_{P_0})$, and similarly $P'_0 \cap f^{-1}(\partial \lceil p \rceil) = f_0^{-1}(\partial \lceil p \rceil_{P_0})$. So h_p sends $|f_0^{-1}(\lceil p \rceil_{P_0})|$ onto $C|f_0^{-1}(\partial \lceil p \rceil_{P_0})|$ and keeps $|f_0^{-1}(\partial \lceil p \rceil_{P_0})|$ fixed. \square

Similarly to Theorem 3.4 one has

Lemma 3.10. *Suppose that we are given monotone maps $f: P' \rightarrow P$, $\pi: P \rightarrow B$ and $\pi': P' \rightarrow B$ [resp. $\pi': P' \rightarrow B^*$] such that f sends $(\pi')^{-1}(\lfloor b \rfloor)$ [resp. $(\pi')^{-1}(\lceil b^* \rceil)$] onto $\pi^{-1}(\lfloor b \rfloor)$ for each $b \in B$. Then f is a fiberwise subdivision map with respect to π and π' if and only if there exists a homeomorphism $h: |P'| \rightarrow |P|$ sending $|f^{-1}(\lceil p \rceil)|$ onto $|p|$ for each $p \in P$ and $|(\pi')^{-1}(\lfloor b \rfloor)|$ [resp. $|(\pi')^{-1}(\lceil b^* \rceil)|$] onto $|\pi^{-1}(\lfloor b \rfloor)|$ for each $b \in B$.*

Corollary 3.11. *Suppose that we are given monotone maps $P'' \xrightarrow{f} P' \xrightarrow{g} P \xrightarrow{\pi} B$, $\pi': P' \rightarrow B$ and $\pi'': P'' \rightarrow B$ [resp. $\pi'': P'' \rightarrow B^*$] such that f sends $(\pi'')^{-1}(\lfloor b \rfloor)$ [resp. $(\pi'')^{-1}(\lceil b^* \rceil)$] onto $(\pi')^{-1}(\lfloor b \rfloor)$ and g sends $(\pi')^{-1}(\lfloor b \rfloor)$ onto $\pi^{-1}(\lfloor b \rfloor)$ for each $b \in B$.*

Then g is a fiberwise subdivision map with respect to π and π' if and only if gf is a fiberwise subdivision map with respect to π and π'' .

To compare combinatorial and topological mapping cylinders it is convenient to work in concrete categories (see [2; 8.10]).

Lemma 3.12. *Let $f: P \rightarrow Q$ be a quotient map in the concrete category of preposets and monotone maps over the category of sets. Then $|f|: |P| \rightarrow |Q|$ is a quotient map in the concrete category of compact polyhedra and PL maps over the category of sets.*

Proof. Clearly, $|f|$ is a quotient map in the concrete category of affine simplicial complexes and affine simplicial maps over the category of sets. This reduces the assertion to a standard fact in PL topology. \square

Lemma 3.13. *For any monotone map $f: P \rightarrow Q$ between posets there exist homeomorphisms $|MC(f)| \cong |MC^*(f)| \cong MC(|f|)$ keeping $|P|$ and $|Q|$ fixed.*

Proof. The subdivision map $I \rightarrow [2]$ yields a subdivision map $\alpha: P \times I \rightarrow P \times [2]$. The monotone involution of I yields a monotone involution h of $P \times I$. Using h_α (see Theorem 3.4) and $|h|$ we obtain a homeomorphism $|P \times I| \cong |P \times [2]|$ and an involution H of $|P \times [2]|$ interchanging $|P \times \{1\}|$ and $|P \times \{2\}|$. These descend to homeomorphisms between $|P \times I| \cup_{|f_0|} |Q|$, $|P \times [2]| \cup_{|f_1|} |Q|$ and $|P \times [2]| \cup_{|f_2|} |Q|$, where f_0 is the composition $P \times \{\{0\}\} \simeq P \xrightarrow{f} Q$. The assertion now follows from Lemma 3.12. \square

Lemma 3.14. *Consider a commutative diagram of monotone maps*

$$\begin{array}{ccc} P' & \xrightarrow{f'} & Q' \\ \downarrow \alpha & \searrow f^\circ & \downarrow \beta \\ P & \xrightarrow{f} & Q, \end{array}$$

where α and β are subdivision maps and f and f' are open maps.

(a) *The induced monotone map $\gamma: MC^*(f^\circ) \rightarrow MC^*(f)$ is a subdivision map over [2].*

(b) *Suppose additionally that there exists a commutative diagram*

$$\begin{array}{ccc} |P'| & \xrightarrow{|f'|} & |Q'| \\ \downarrow h_\alpha & & \downarrow h_\beta \\ |P| & \xrightarrow{|f|} & |Q|, \end{array}$$

where h_α and h_β are underlying homeomorphisms of the subdivision maps α and β . Then the induced monotone map $\delta: MC^*(f') \rightarrow MC^*(f^\circ)$ is a subdivision map over [2].

Note that the upper-right triangle is irrelevant for the purposes of (a), since one can take $\beta = \alpha$ and $f' = f^\circ$.

A diagram as above except that f and f' are not assumed to be open can be converted into a diagram as above by using the handle functor, see Theorem 3.68 below.

Proof. Since α is open, so is f° . Hence $M := MC^*(f)$, $M' := MC^*(f')$ and $M^\circ := MC^*(f^\circ)$ are posets. Let us identify P , P' , Q and Q' with their copies in M , M' and M° . Then $\delta = \text{id}_{P'} \cup \beta$ and $\gamma = \alpha \cup \text{id}_Q$.

(a) Given a $p \in P$, $\lceil p \rceil_M = \lceil p \rceil_P$. Then $\gamma^{-1}(\lceil p \rceil_M) = \alpha^{-1}(\lceil p \rceil_P)$ and $\gamma^{-1}(\partial \lceil p \rceil_M) = \alpha^{-1}(\partial \lceil p \rceil_P)$. Hence γ is a subdivision map over P .

Given a $q \in Q$, we have $\gamma^{-1}(\lceil q \rceil_M) = \lceil q \rceil_{M^\circ}$ and $\gamma^{-1}(\partial \lceil q \rceil_M) = \partial \lceil q \rceil_{M^\circ}$. Thus γ is a subdivision map. Moreover, it is a subdivision map over [2] since $\gamma|_Q = \text{id}_Q$.

(b) Clearly δ is the identity (and hence a subdivision map) over P' .

Now let us fix a $q \in Q$. Let us write $B = \lceil q \rceil_Q$, $A = f^{-1}(B)$ and $dA = f^{-1}(\partial B)$. Let $B' = \beta^{-1}(B)$, so that $\partial B' = \beta^{-1}(\partial B)$. Further let $A' = \alpha^{-1}(A) = (f^\circ)^{-1}(B)$ and $dA' = \alpha^{-1}(dA) = (f^\circ)^{-1}(\partial B)$.

We have $\lceil q \rceil_M = MC^*(A \xrightarrow{f|} B)$ and $\partial \lceil q \rceil_M = A \cup MC^*(dA \xrightarrow{f|} \partial B)$. Similarly, $\lceil q \rceil_{M^\circ} = MC^*(A' \xrightarrow{f^\circ|} B)$ and $\partial \lceil q \rceil_{M^\circ} = A' \cup MC^*(dA' \xrightarrow{f^\circ|} \partial B)$. Hence $\delta^{-1}(\lceil q \rceil_{M^\circ}) = MC^*(A' \xrightarrow{f'|} B')$ and $\delta^{-1}(\partial \lceil q \rceil_{M^\circ}) = A' \cup MC^*(dA' \xrightarrow{f'|} \partial B')$.

By the hypothesis $|f'|$ is conjugate to $|f|$, namely $|f'| = h_\beta^{-1}|f|h_\alpha$. Since h_α and h_β are underlying homeomorphisms of α and β , they also carry $|A'| \xrightarrow{|f'|} |B'|$ onto $|A| \xrightarrow{|f|} |B|$, and $|dA'| \xrightarrow{|f'|} |\partial B'|$ onto $|dA| \xrightarrow{|f|} |\partial B|$. Since $|MC^*(\varphi)| \cong MC(|\varphi|)$ by

Lemma 3.13, it follows that $(|MC^*(A' \xrightarrow{f'} B')|, |A' \cup MC^*(dA' \xrightarrow{f'} \partial B')|)$ is homeomorphic to $(|MC^*(A \xrightarrow{f} B)|, |A \cup MC^*(dA \xrightarrow{f} \partial B)|)$. But the latter pair is the same as $(|\lceil q \rceil_M|, |\partial \lceil q \rceil_M|)$. \square

In the case $\beta = \alpha$, $f = \text{id}$, $f' = \text{id}$ we get

Corollary 3.15. *Let $\alpha: K' \rightarrow K$ be a subdivision map. Then the induced monotone maps $MC^*(\text{id}_{K'}) \xrightarrow{\delta} MC^*(\alpha) \xrightarrow{\gamma} MC^*(\text{id}_K)$ are subdivision maps over [2].*

3.B. Collaring

In this subsection we give a new short proof of Whitehead's collaring theorem for simplicial complexes (see [57] and [21; 4.2] for two other proofs and references to older proofs) using mapping cylinders of non-simplicial maps. We also note the (rather trivial) generalization to posets, which will be used later.

3.16. Collaring. We say that a closed subposet Q of P is *collared* in P if for every $q \in Q$, $|\text{lk}(q, P)|$ is homeomorphic to the cone over $|\text{lk}(q, Q)|$ by a homeomorphism keeping $|\text{lk}(q, Q)|$ fixed.

Lemma 3.17. *If Q is a closed subposet of a poset P , the property of Q to be collared in P is an invariant of the homeomorphism type of the pair $(|P|, |Q|)$*

Proof. Given a $q \in Q$, let q_- be a maximal simplex of $\lceil q \rceil^b$. Then $\text{lk}(q, P)^b = \text{Lk}(q_-, P^b)$ as subcomplexes of P^b , and similarly $\text{lk}(q, Q)^b = \text{Lk}(q_-, Q^b)$. Also $(\text{Lk}(q_-, P^b), \text{Lk}(q_-, Q^b))$ is isomorphic to $(\text{lk}(q_-, P^b), \text{lk}(q_-, Q^b))$. Since every simplex of Q^b is a maximal simplex of $\lceil q \rceil^b$ for some $q \in Q$, we get that Q is collared in P if and only if Q^b is collared in P^b .

This reduces the lemma to the case where P is a simplicial complex. This case follows by the combinatorial invariance of link. \square

If Q is a closed subposet of a poset P , let P_Q^+ denote the amalgam $P \cup_{Q=Q \times \{0\}} Q \times I$.

Lemma 3.18. *Let P be a simplicial complex and Q a full subcomplex of P . Then Q is collared in P if and only if there exists a homeomorphism $h: |P| \rightarrow |P^+|$ keeping $|P \setminus \lceil Q \rceil|$ fixed and extending the obvious homeomorphism $|Q| \rightarrow |Q \times \{1\}|$. Furthermore, h sends $|\lceil Q \rceil|$ onto $|Q \times I|$ and $|P \setminus Q|$ onto $|P|$.*

Proof. We first note that $Q \times \{1\}$ is collared in $Q \times I$. Indeed, given a $q \in Q$, we have $\text{lk}((q, 1), Q \times I) \simeq \text{lk}(q, Q) * \text{lk}(\{1\}, I) \simeq \text{lk}(q, Q) * pt$, and this isomorphism sends $\text{lk}((q, 1), Q \times \{1\})$ onto $\text{lk}(q, Q) * \emptyset$. Hence $Q \times \{1\}$ is collared in P_Q^+ . The “if” implication now follows from Lemma 3.17.

Conversely, suppose that Q is collared in P . Let $Q' = \lceil Q \rceil \setminus Q$ and let $R = \lceil \lceil Q \rceil \rceil \setminus \lceil Q \rceil$. Since R is closed in P , it is a simplicial complex. (In fact, Q' is a cubosimplicial complex, see Lemma 2.42.) Since Q is full in P , for each $p \in Q'$ the simplex $\lceil p \rceil$ meets Q in a simplex $\lceil q_p \rceil$. Then $\lceil p \rceil$ meets R in the opposite simplex $\lceil r_p \rceil = \lceil p \rceil \setminus \lceil q_p \rceil$. The resulting maps $f: Q' \rightarrow Q$, $p \mapsto q_p$, and $g: Q' \rightarrow R$, $p \mapsto r_p$, are easily seen to be monotone and

(using that P is a simplicial complex) closed. (In fact, they are cubical, see Lemma 2.46.) It follows that $\lceil Q' \rceil$ is isomorphic to the double mapping cylinder $MC(f) \cup_{Q'} MC(g)$. Note that $P = \lceil Q' \rceil \cup (P \setminus \lfloor Q \rfloor)$ and $\lceil Q' \rceil \cap (P \setminus \lfloor Q \rfloor) = R$.

Pick some $q \in Q$. Let $K = \text{lk}(q, P) \setminus \text{lk}(q, Q)$ and let $L = \lfloor \text{lk}(q, Q) \rfloor_{\text{lk}(q, P)} \setminus \text{lk}(q, Q)$. Clearly, $f^{-1}(\lfloor q \rfloor) = K$ and $f^{-1}(\partial^* \lfloor q \rfloor) = L$.

Since Q is collared in P , clearly $\text{lk}(q, Q)$ is collared in $\text{lk}(q, P)$. On the other hand, $\text{Lk}(q, Q)$ is full in $\text{Lk}(q, P)$ since Q is full in P , and $\text{Lk}(q, P)$ is a simplicial complex since P is. Since $\text{lk}(q, Q)$ is naturally isomorphic to $\text{Lk}(q, Q)$, we get that $\text{lk}(q, Q)$ is full in $\text{lk}(q, P)$, which is a simplicial complex. Arguing by induction, we may assume that there is a homeomorphism $|\text{lk}(q, P)| \rightarrow |\text{lk}(q, P)_{\text{lk}(q, Q)}^+|$ which sends $|K|$ onto $|\text{lk}(q, P)|$ and $|L|$ onto $|\text{lk}(q, Q)|$. Hence $(|K|, |L|) \cong (|\text{lk}(q, P)|, |\text{lk}(q, Q)|)$. Since Q is collared in P , we get that $|K| \cong C|L|$ keeping $|L|$ fixed. Thus f^* is a subdivision map.

Since $MC^*(f^*) \simeq MC(f)^*$, we get using Corollary 3.15 that the duals of the natural monotone maps $MC(\text{id}_{Q'}) \xrightarrow{\varphi} MC(f) \xrightarrow{\psi} MC(\text{id}_Q)$ are subdivision maps. On the other hand, the projection $Q' \times (I^*) \rightarrow Q'$ extends to a subdivision map $MC(f) \cup_{Q'} Q' \times (I^*) \cup_{Q'} MC(g) \rightarrow MC(f) \cup_{Q'} MC(g)$. By Theorem 3.4, this yields a homeomorphism $|MC(f) \cup_{Q'} Q' \times (I^*) \cup_{Q'} MC(g)| \cong |MC(f) \cup_{Q'} MC(g)|$. We note that if X is a poset, $X \times I$ is the amalgam of two copies of $MC(\text{id}_X)$ along the two open copies of X , and $X \times (I^*)$ is that along the two closed copies of X (see Lemma 2.62). Hence φ , $f \times \text{id}_{[2]}$ and ψ combine into a monotone map $Q' \times (I^*) \cup_{Q'} MC(f) \rightarrow MC(f) \cup_Q Q \times I$ whose dual is a subdivision map. By Theorem 3.4 this yields a homeomorphism $|Q' \times (I^*) \cup_{Q'} MC(f)| \cong |MC(f) \cup_Q Q \times I|$. Combining it with the previous homeomorphism, we obtain the desired homeomorphism $|P| \cong |P_Q^+|$. \square

Theorem 3.19. *Let P be a poset and Q a closed subposet of P . Then Q is collared in P if and only if there exists a homeomorphism $h: |P| \rightarrow |P_Q^+|$ keeping $|P \setminus \lfloor Q \rfloor|$ fixed and extending the homeomorphism $|Q| \rightarrow |Q \times \{1\}|$.*

Proof. By Lemma 3.17 we may replace (P, Q) by (P^b, Q^b) . Since Q^b is full in P^b , the assertion follows from Lemma 3.18. \square

Addendum 3.20. *Let P be a poset and Q a collared closed subposet of P . Further let R be an open subposet of P , and write $S = R \cap Q$. Then the homeomorphism h of Theorem 3.19 sends $|R|$ onto $|R_S^+|$ and extends the homeomorphism $|S| \rightarrow |S \times \{1\}|$.*

Proof. Since R is open in P , $\text{lk}(r, R) = \text{lk}(r, P)$ and $\text{lk}(r, S) = \text{lk}(r, Q)$ for each $r \in S$; in particular, S is collared in R . It follows that for each $\sigma \in S^b$, $(|\text{lk}(\sigma, P^b)|, |\text{lk}(\sigma, R^b)|)$ is homeomorphic to the cone over $(|\text{lk}(\sigma, Q^b)|, |\text{lk}(\sigma, S^b)|)$ keeping $(|\text{lk}(\sigma, Q^b)|, |\text{lk}(\sigma, S^b)|)$ fixed. The proof of Lemma 3.18 then works to show that the subdivision map f^* defined in there is fiberwise with respect to $\chi: (Q^b)^* \rightarrow [2]$, where $\chi^{-1}(2) = (S^b)^*$. The remainder of the proof closely follows that of Theorem 3.19. \square

Lemma 3.21. *If P and Q are posets, and P_0 is a closed subposet of P such that $|Q + P|$ is homeomorphic to $|C(Q + P_0)|$ keeping $|Q + P_0|$ fixed, then $|P|$ is homeomorphic to $|CP_0|$ keeping P_0 fixed.*

Proof. Since $Q + P_0$ is collared in $C(Q + P_0)$, by Lemma 3.17 it is also collared in $Q + P$. Thus $|\text{lk}(q, Q + P)| \cong |C \text{lk}(q, Q + P_0)|$ keeping $|\text{lk}(q, Q + P_0)|$ fixed for each $q \in Q$. On the other hand, if q is a maximal element of Q , then $(\text{lk}(q, Q + P), \text{lk}(q, Q + P_0)) \simeq (P, P_0)$. \square

Remark 3.22. The Armstrong–Morton argument (see [8; p. 180]) works to show that if (P, Q) and (X, Y) are pairs of polyhedra such that $(S^n * P, S^n * Q)$ is homeomorphic to $(S^n * X, S^n * Y)$, then (P, Q) is homeomorphic to (X, Y) . (Beware that (P, Q) need not be a suspension if P and Q are suspensions, as shown by classical knots.) Similarly, Morton’s argument [52; Theorem 2] works to show that any pair (P, Q) of polyhedra factors uniquely as a ball or sphere joined to a pair of polyhedra that is reduced in the sense that it is neither a cone pair nor a suspension pair. A relative version of Morton’s uniqueness of factorization of reduced polyhedra into joins (which would imply Lemma 3.21) could be more tricky, however.

Lemma 3.23. *Let P be a poset and Q, R its closed subposets such that $P = Q \cup R$, where $Q \cap R$ is collared in Q and in R . Then*

- (a) Q is collared in CP ;
- (b) $(Q \cap R) \times CZ$ and $(Q \cap R) \times CZ \cup Q \times Z$ are collared in $R \times CZ \cup CP \times Z$ for every poset Z .

Proof. (a). Given a $p \in Q$, we need to show that $|\text{lk}(p, CP)|$ is homeomorphic to $C|\text{lk}(p, Q)|$ keeping $|\text{lk}(p, Q)|$ fixed. If $p \notin Q \cap R$, then $\text{lk}(p, Q) = \text{lk}(p, P)$, and the assertion follows since P is collared in CP . Suppose that $p \in Q \cap R$. Then $|\text{lk}(p, P)|$ is homeomorphic to $\{q, r\} * |\text{lk}(p, Q \cap R)|$ keeping $|\text{lk}(p, Q \cap R)|$ fixed, and this extends to a homeomorphism between $|\text{lk}(p, CP)|$ and $\{x\} * \{q, r\} * |\text{lk}(p, Q \cap R)|$. Now $\{x\} * \{q, r\}$ is homeomorphic to $C\{r\}$ keeping $\{r\}$ fixed, which implies the assertion. \square

(b). Similarly to the proof of (a), the assertion follows from the fact that $\{x\} * \{q, r\} \cup_{\{r\}} \{r\} * \{y\}$ is homeomorphic to $C\{q\}$ keeping $\{q\}$ fixed and to $C\{q, y\}$ keeping $\{q, y\}$ fixed. \square

3.C. Stratification maps

3.24. Stratification map. We say that $f: P \rightarrow Q$ is a *stratification map* if $f^{-1}(\partial[q])$ is collared in $f^{-1}(\lceil q \rceil)$ for each $q \in Q$. When Q is a cell complex, stratification map is a combinatorial form of the notion of a “polyhedral mock bundle” of [18; p. 35].

Clearly, every subdivision map is a stratification map, and every stratification map is an open map. If $f: P \rightarrow Q$ is a stratification map, then by the combinatorial invariance of $\text{link } \partial P = f^{-1}(\partial Q)$.

Theorem 3.25. *A monotone map of posets $f: P \rightarrow Q$ is a stratification map if and only if for each $p \in P$ the map*

$$f_p: \text{lk}(p, P) \rightarrow F + \text{lk}(f(p), Q),$$

defined by the identity on $F := \text{lk}(p, f^{-1}(f(p)))$ and as the restriction of f elsewhere, is a subdivision map.

Proof. Let us write $\text{lk } p = \text{lk}(p, P)$ and $\text{lk } f(p) = \text{lk}(f(p), Q)$. We observe that the composition of f_p and the “projection” $h: F + \text{lk } f(p) \rightarrow C^* \text{lk } f(p) \simeq \lfloor f(p) \rfloor$ coincides with $f|_{\text{lk } p}$. Pick some $q \in \text{lk } f(p)$. Let us consider $M_{pq} := f_p^{-1}(\lceil q \rceil_{F + \text{lk } f(p)})$ and $L_{pq} := f_p^{-1}(\partial \lceil q \rceil_{F + \text{lk } f(p)})$. The cone $\lceil q \rceil_{F + \text{lk } f(p)} = F + \lceil q \rceil_{\text{lk } f(p)}$, whence $M_{pq} = g^{-1}(h^{-1}(\lceil q \rceil_{C^* \text{lk } f(p)})) = (f|_{\text{lk } p})^{-1}(\lceil q \rceil_Q)$; similarly, $L_{pq} = (f|_{\text{lk } p})^{-1}(\partial \lceil q \rceil_Q)$. Let us consider $M_q := f^{-1}(\lceil q \rceil_Q)$ and $L_q := f^{-1}(\partial \lceil q \rceil_Q)$. Then $M_{pq} = \text{lk}(p, M_q)$ and $L_{pq} = \text{lk}(p, L_q)$.

If L_q is collared in M_q , then $|M_{pq}| \cong C|L_{pq}|$ keeping $|L_{pq}|$ fixed. This works for every $q \in \text{lk } f(p)$; so if f is a stratification map, then f_p is a subdivision map.

Conversely, if f_p is a subdivision map, then $|M_{pq}| \cong C|L_{pq}|$ keeping $|L_{pq}|$ fixed. Hence if f_p is a subdivision map for every $p \in f^{-1}(\partial \lceil q \rceil)$, then L_q is collared in M_q . This works for every $q \in Q$; so if f_p is a subdivision map for all $p \in P$, then f is a stratification map. \square

Corollary 3.26. *A cubical map of posets is dual to a stratification map.*

Proof. If $f^*: P^* \rightarrow Q^*$ is cubical, the restriction $\lfloor p \rfloor \rightarrow \lfloor f(p) \rfloor$ of f is isomorphic to a projection of the form $CX \times CY \rightarrow CY$ for each $p \in P$. Then f_p is isomorphic to the monotone map $X * Y \rightarrow X + Y$ that restricts to the identity map $X * \emptyset \rightarrow X$ and projects $C^*X \times Y$ onto Y . Given a $y \in Y$, we have $\lceil y \rceil_{X+Y} = X + \lceil y \rceil_Y$, and consequently $f^{-1}(\lceil y \rceil) = X * \lceil y \rceil_Y$ and $f^{-1}(\partial \lceil y \rceil) = X * \partial \lceil y \rceil_Y$. Let $Z = \partial \lceil y \rceil_Y$. By the associativity of join of polyhedra, $|X * CZ| \cong C|X * Z|$ keeping $|X * Z|$ fixed. \square

Theorem 3.27. *Let $f: P \rightarrow Q$ be a stratification map and Z a closed subposet of Q .*

- (a) *If Z is collared in Q , then $f^{-1}(Z)$ is collared in P .*
- (b) *When $Z \subset f(P)$, the converse holds.*

Proof. (a). Let $p \in Y := f^{-1}(Z)$, and write $F = \text{lk}(p, f^{-1}(f(p)))$. By Theorem 3.25, $f_p: \text{lk}(p, P) \rightarrow F + \text{lk}(f(p), Q)$ is a subdivision map. Then by Theorem 3.4 there is a homeomorphism $|\text{lk}(p, P)| \rightarrow |F + \text{lk}(f(p), Q)|$, which takes $|\text{lk}(p, Y)|$ onto $|F + \text{lk}(f(p), Z)|$. Hence $|\text{lk}(p, P)| \cong C|\text{lk}(p, Y)|$ keeping $|\text{lk}(p, Y)|$ fixed if and only if $|F + \text{lk}(f(p), Q)| \cong C|F + \text{lk}(f(p), Z)|$ keeping $|F + \text{lk}(f(p), Z)|$ fixed. The latter follows from the existence of a homeomorphism $|\text{lk}(f(p), Q)| \cong C|\text{lk}(f(p), Z)|$ keeping $|\text{lk}(f(p), Z)|$ fixed. \square

(b). By Lemma 3.21, the last inference in the proof of (a) can be reversed. \square

Corollary 3.28. *Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be monotone maps of posets. If f and g are stratification maps, then so is gf . If f and gf are stratification maps, and f is surjective, then g is a stratification map.*

Lemma 3.29. *Let P and Q be posets. Then*

- (a) $\partial(P \times Q) = (\partial P) \times Q \cup P \times (\partial Q)$;
- (b) *if ∂P is collared in P and ∂Q is collared in Q , then $\partial(P \times Q)$ is collared in $P \times Q$.*

Part (a), which is easy, implies in particular that $CA \times CB \simeq C(A \times CB \cup CA \times B)$.

Proof of (b). We recall the formula $\text{lk}((p, q), P \times Q) \simeq \text{lk}(p, P) * \text{lk}(q, Q)$. Let (p, q) be an element of $\partial(P \times Q)$. We want to show that $|\text{lk}(p, P) * \text{lk}(q, Q)| \cong C|\text{lk}((p, q), \partial(P \times Q))|$ keeping $|\text{lk}((p, q), \partial(P \times Q))|$ fixed.

By symmetry we may assume that $p \in \partial P$. Then $|\text{lk}(p, P)| \cong C|\text{lk}(p, \partial P)|$ keeping $|\text{lk}(p, \partial P)|$ fixed. Hence $|\text{lk}(p, P) * \text{lk}(q, Q)| \cong C|\text{lk}(p, \partial P) * \text{lk}(q, Q)|$ keeping $|\text{lk}(p, \partial P) * \text{lk}(q, Q)|$ fixed. Indeed, $(CX) * Y \cong C(X * Y)$ keeping $X * Y$ fixed, due to $(c * A) * B \simeq c * (A * B)$.

If $q \notin \partial Q$, then $\text{lk}((p, q), \partial(P \times Q)) = \text{lk}((p, q), (\partial P) \times Q)$ and the assertion follows.

If $q \in \partial Q$, then $|\text{lk}(q, Q)| \cong C|\text{lk}(q, \partial Q)|$ keeping $|\text{lk}(q, \partial Q)|$ fixed. Also

$$\text{lk}((p, q), \partial(P \times Q)) = \text{lk}((p, q), (\partial P) \times Q) \bigcup_{\text{lk}((p, q), (\partial P) \times (\partial Q))} \text{lk}((p, q), P \times \partial Q).$$

The assertion now follows using that $(CX) * (CY) \cong C(CX * Y \cup_{X * Y} X * CY)$ keeping $CX * Y \cup_{X * Y} X * CY$ fixed, which in turn is a consequence of Lemma 2.37. \square

Corollary 3.30. *Let $f: P \rightarrow Q$ and $g: R \rightarrow S$ be stratification maps or subdivision maps. Then so is $f \times g: P \times R \rightarrow Q \times S$.*

Lemma 3.31. *Let Q be an open subposet of a poset P .*

(a) *The inclusion $Q \hookrightarrow P$ is a stratification map.*

(b) *A monotone map $f: R \rightarrow Q$ is a stratification map if and only if the composition $R \xrightarrow{f} Q \hookrightarrow P$ is.*

(c) *If $g: S \rightarrow P$ is a stratification map, then so is $g|: g^{-1}(Q) \rightarrow Q$.*

Proof. (a). Given a $p \in P$, $\partial[p]_Q$ is the boundary of the cone $[p]_Q$, and therefore is collared in the latter. \square

(b). The “if” implication is straightforward from the definition, and the “only if” implication follows from Corollary 3.28. \square

(c). Since Q is open in P , $g^{-1}(Q)$ is open in S . Then by (a) and Corollary 3.28, the composition $g^{-1}(Q) \hookrightarrow S \xrightarrow{g} P$ is a stratification map. Then by (b) so is the restriction $g|: g^{-1}(Q) \rightarrow Q$. \square

3.D. Filtration maps

3.32. Codimension one. We say that a closed subposet Q of P is *of codimension one* in P if every maximal element of Q is covered by at least one maximal element of P . If additionally no maximal element of Q is covered by a non-maximal element of P , then we say that Q is *of pure codimension one* in P .

We note that if Q is collared in P , then the link in P of every maximal element of Q is a singleton; in other words, every maximal element of Q is covered by *precisely* one maximal element of P , and by no other element of P . On the other hand, if Q is of codimension one in P , then it is nowhere dense in P , in the sense of the Alexandroff topology (i.e. contains no maximal element of P).

3.33. Filtration map. We call a monotone map of posets $f: P \rightarrow Q$ a *[pure] filtration map* if $f^{-1}(\partial \lceil q \rceil)$ is of [pure] codimension one in $f^{-1}(\lceil q \rceil)$ for each $q \in Q$. Filtration maps (an in particular stratification maps) are open, since if $q > f(p)$, then $f^{-1}(\partial \lceil q \rceil)$ contains a maximal element p' such that $p' \geq p$, and this p' must be covered by an element of $f^{-1}(\lceil q \rceil)$, which can only be in $f^{-1}(q)$.

Lemma 3.34. *Let $f: P \rightarrow Q$ be a filtration map and Z a closed subposet of Q . If Z is of codimension one in Q , then $f^{-1}(Z)$ is of codimension one in P .*

Proof. Let p be a maximal element of $f^{-1}(Z)$. The restriction of f to $f^{-1}(Z)$ is a filtration map, and in particular an open map. Hence $f(\{p\})$ is open in Z , that is, $f(p)$ is maximal in Z . Since Z is of codimension one in Q , $f(p)$ is covered by a maximal element q of Q . Since $f(p)$ is maximal in Z , $q \notin Z$.

Since $q > f(p)$, we have $p \in f^{-1}(\partial \lceil q \rceil)$. Let p' be a maximal element of $f^{-1}(\partial \lceil q \rceil)$ such that $p' \geq p$. Since f is a filtration map, p' is covered by a maximal element p'' of $f^{-1}(\lceil q \rceil)$. Since p' is maximal in $f^{-1}(\partial \lceil q \rceil)$, $p'' \notin f^{-1}(\partial \lceil q \rceil)$, that is, $f(p'') = q$.

If $p' > p$, then since p is maximal in $f^{-1}(Z)$, $p' \notin f^{-1}(Z)$, that is, $f(p') \notin Z$. Since $f(p) \in Z$, $f(p') \neq f(p)$. On the other hand, $p' \in f^{-1}(\partial \lceil q \rceil)$, so $f(p') \neq q$. Hence $f(p) < f(p') < q$, contradicting our choice of q . Thus $p' = p$, and so p'' covers p .

If $p''' \geq p''$, then $f(p''') = f(p'')$ since $f(p'') = q$ is maximal in Q . Hence $p''' \in f^{-1}(\lceil q \rceil)$, and so $p''' = p''$ since p'' is maximal in $f^{-1}(\lceil q \rceil)$. Thus p'' is maximal in P . \square

Corollary 3.35. *Filtration maps are closed under composition.*

Example 3.36. Let $P = (\{a, b, c, d\}, \leq)$, where the only relations are $a < b$ and $a < c < d$ and their implications. Let Q be the subposet of P formed by a, c and d , and let $f: P \rightarrow Q$ be the retraction sending b onto d . Then f is a filtration map, and $\{a\}$ is of codimension one in P but not in Q .

Theorem 3.37. *Let $f: P \rightarrow Q$ be a pure filtration map and Z a closed subposet of Q .*

(a) *If Z is of pure codimension one in Q , then $f^{-1}(Z)$ is of pure codimension one in P .*

(b) *When $Z \subset f(P)$, the converse holds.*

Proof. (a). Let p be a maximal element of $f^{-1}(Z)$. By Lemma 3.34, p is covered by a maximal element of P . Suppose that p is also covered by a $p' \in P$, which in turn is covered by a $p'' \in P$. The restriction of f to $f^{-1}(Z)$ is a pure filtration map, and in particular an open map. Hence $f(\{p\})$ is open in Z , that is, $f(p)$ is maximal in Z .

Since p is maximal in $f^{-1}(Z)$, $p' \notin f^{-1}(Z)$, that is, $f(p') \notin Z$. Since $f(p) \in Z$, $f(p') \neq f(p)$. Thus $f(p') > f(p)$; since Z is of pure codimension one in Q , $f(p')$ covers $f(p)$.

Suppose that $f(p'') = f(p')$. Then p' and p'' lie in $f^{-1}(R)$, where $R = \lceil f(p'') \rceil$. If $p''' \in f^{-1}(\partial R)$ and $p''' \geq p$, then $f(p''') = f(p)$ since $f(p)$ is maximal in ∂R . Then $p''' \in f^{-1}(Z)$, so $p''' = p$ since p is maximal in $f^{-1}(Z)$. Thus p is maximal in $f^{-1}(\partial R)$. Then the inequalities $p < p' < p''$ contradict the hypothesis that f is a pure filtration

map. Thus $f(p'') \neq f(p')$. Then the inequalities $f(p'') > f(p') > f(p)$ contradict the hypothesis that Z is of pure codimension one in Q . \square

(b). Let q be a maximal element of Z . Since $Z \subset f(P)$, there exists a maximal element p in $f^{-1}(q)$. Since $f^{-1}(q)$ is open in $f^{-1}(Z)$, p is maximal in $f^{-1}(Z)$. Since $f^{-1}(Z)$ is of codimension one in P , p is covered by a maximal element p' of P . Since p is maximal in $f^{-1}(Z)$, $p' \notin f^{-1}(Z)$, that is, $f(p') \notin Z$. Since $p' \geq p$, $f(p') \geq f(p) = q$. Since $q \in Z$, $f(p') > q$. Since Z is of pure codimension one in Q , $f(p')$ covers q . Since f is open and p' is maximal in P , $f(p')$ is maximal in Q .

Suppose that q is covered by a $q' \in Q$, which is in turn covered by a $q'' \in Q$. Since f is open, $q' = f(p'')$ for some $p'' > p$. By the same token $f(p') = f(p''')$ for some $p''' > p''$. Since p is maximal in $f^{-1}(Z)$, the inequalities $p < p'' < p'''$ contradict the hypothesis that $f^{-1}(Z)$ is of pure codimension one in P . \square

Corollary 3.38. *Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be monotone maps of posets. If f and g are pure filtration maps, then so is gf . If f and gf are pure filtration maps, and f is surjective, then g is a pure filtration map.*

3.E. Manifolds

3.39. Sphere and ball. We call a poset P an n -sphere if $|P|$ is homeomorphic to $|\partial\Delta^{n+1}|$, and an n -ball with boundary ∂P if ∂P is a closed subposet of P and $(|P|, |\partial P|)$ is homeomorphic to $(|\Delta^n|, |\partial\Delta^n|)$.

We recall that a *PL manifold* is polyhedron M where each point lies in the interior of a closed subpolyhedron PL homeomorphic to a simplex; the *boundary* of M consists of those points that are taken into the boundary of the simplex. It is well-known (see [41]) that, given a simplicial complex K and its subcomplex L , $|K|$ is a PL n -manifold with boundary $|L|$ if and only if K is a *combinatorial n -manifold* with boundary L . The latter means that for every i -simplex $\sigma \in K$, the link $\text{Lk}(\sigma, K)$ is an $(n - i - 1)$ -sphere if $\sigma \notin \partial K$ and an $(n - i - 1)$ -ball with boundary $\text{Lk}(\sigma, \partial K)$ if $\sigma \in \partial K$. It is easy to see that the case $i = 0$ of this condition implies the other cases.

3.40. Manifold. We call a poset P a *manifold with [co]boundary ∂P* if ∂P is a (possibly empty) closed [resp. open] subposet of P , and $|P|$ is a PL manifold with boundary $|\partial P|$. Clearly, P is a manifold with boundary Q if and only if P^* is a manifold with coboundary Q^* .

3.41. Cell complex. A poset P is called a *cell complex*, if $\partial[p]$ is a sphere of some dimension for each $p \in P$. The cones of a cell complex are also referred to as its *cells*; clearly, they are balls. A closed subposet of a cell complex will be referred to as its *subcomplex*.

Affine polytopal complexes are clearly cell complexes.

Lemma 3.42. *Let P be a poset. Then $P^\#$ is a cell complex if and only if the open interval $\partial\partial^*[p, q]$ is a sphere for every $p, q \in P$ with $p < q$.*

Proof. If $S := \partial\partial^*[p, q]$ is an n -sphere, then $B_1 := (\partial[p, q])^\#$ and $B_2 := (\partial^*[p, q])^\#$ are $(n+1)$ -balls with common boundary $S^\#$, so $B_1 \cup B_2 = \partial([p, q]^\#)$ is an $(n+1)$ -sphere.

Conversely, suppose that $B_1 \cup B_2$ is an m -sphere. Since each $|B_i| \cong C|\partial B_i|$ keeping $|\partial B_i|$ fixed, we have $|B_1 \cup B_2| \cong S^0 * |S|$. Hence by [52] $|S|$ is an $(m-1)$ -sphere. \square

3.43. Cell complex with coboundary. A poset P will be called a *cell complex with coboundary* ∂^*P , if ∂^*P is an open subposet of P , and $(\partial[p])^*$ is a sphere for each $p \in P \setminus \partial^*P$ and a ball with boundary $(\partial[p]_{\partial^*P})^*$ for each $p \in \partial^*P$.

If P is a cell complex, then C^*P is a cell complex with coboundary P , so the new definition of coboundary ∂^*P extends the former one. A closed subposet K of a cell complex P with coboundary is clearly a cell complex with coboundary $K \cap \partial^*P$.

Remark 3.44. Obviously, a poset P is a cell complex with coboundary Q if and only if P and $P \setminus Q$ are cell complexes, and Q^* is collared in P^* .

3.45. Pseudo-manifold. A poset P is called *purely n -dimensional* if every simplex of P^\flat lies in an n -simplex; and *pure* if it is purely n -dimensional for some n . (Some authors call such posets ranked or graded, but those terms may also mean something else.) P will be called an *n -pseudo-manifold with coboundary Q* if P is purely n -dimensional, P is a cell complex with coboundary Q , and the 1-skeleton $(P^*)^{(1)}$ (which consists of atoms and elements that cover atoms) is a cell complex (in other words, a graph). For brevity, P^* will be called an *n -pseudo*-manifold with boundary Q^** .

It is easy to see that if P is a closed n -pseudo-manifold, then C^*P is an $(n+1)$ -pseudo-manifold with coboundary P . Dually, $C(P^*)$ is an $(n+1)$ -pseudo*-manifold with boundary P^* .

The following basic result is essentially known (see [48; 4.1], where the case $\partial P = \emptyset$ is proved by a nontrivial topological argument) but for completeness we spell out the details of a trivial combinatorial argument.

Theorem 3.46. *Let P be a poset and Q a closed subposet of P . Then P is a manifold with boundary Q if and only if P is pure, P is a cell complex and P^* is a cell complex with coboundary Q^* .*

Proof. Suppose that P is a manifold with boundary ∂P . By definition this implies that the barycentric subdivision P^\flat is a combinatorial manifold with boundary Q^\flat . Given a $p \in P$, let p_- be a maximal simplex of $[p]^\flat$, and p_+ a maximal simplex of $[p]^\flat$. Then $\text{lk}(p, P)^\flat = \text{Lk}(p_-, P^\flat)$ and $\text{lk}(p^*, P^*)^\flat = \text{Lk}(p_+, P^\flat)$ as subcomplexes of P^\flat . When $p \in Q$, the same equations also hold with Q in place of P .

Since Q is closed in P , we have $p_- \in Q^\flat$ if and only if $p \in Q$; whereas $p_+ \notin Q^\flat$ for all $p \in P$. Hence $\text{lk}(p^*, P^*)$ is a sphere for all $p \in P$; whereas $\text{lk}(p, P)$ is a sphere if $p \notin Q$, and a ball with boundary $\text{lk}(p, Q)$ if $p \in Q$. Since $\text{lk}(p^*, P^*) = (\partial[p]_{P^*})^*$ and $\text{lk}(p, P) = (\partial[p]_{P^*})^*$, we conclude that P is a cell complex and P^* is a cell complex with coboundary Q^* .

Conversely, suppose that P is a cell complex and P^* is a cell complex with coboundary Q^* . For each $p \in P$, we have $\text{Lk}(\{p\}, P^\flat) = \text{lk}(p, P)^\flat * \text{lk}(p^*, P^*)^\flat$ as subcomplexes of

P^\flat . When $p \in Q$, the same equation also holds with Q in place of P . Since the join of two PL spheres is a PL sphere, and the join of a PL ball with a PL sphere is a PL ball with appropriate boundary, it follows that for each vertex v of P^\flat , the link $\text{Lk}(v, P^\flat)$ is a sphere if $v \notin Q^\flat$, and a ball with boundary $\text{Lk}(v, Q^\flat)$ if $v \in Q^\flat$.

The hypothesis that P is pure ensures that different connected components of $|P|$ are PL manifolds of the same dimension. \square

Corollary 3.47. *Spheres, balls and other manifolds are cell complexes.*

Since balls are cell complexes, we also get

Corollary 3.48. *A subdivision of a cell complex is a cell complex.*

Lemma 3.49. *A poset P is a cell complex with coboundary Q if and only if $(\partial[p])^*$ is a sphere for all $p \in P \setminus Q$ and a ball for all $p \in Q$.*

Proof. The “only if” direction is trivial. Conversely, by the hypothesis $P \setminus Q$ is a cell complex. So it is closed in P , whence Q is open in P .

Pick some $q \in Q$, and let $B = \partial[q]_P$. Then B is a subcomplex of P , and since B^* is a ball, by Theorem 3.46, B is a cell complex with coboundary. If $p \in B \setminus \partial^*B$, then $\partial[p]_P = \partial[p]_B$ is a sphere, so $p \notin Q$. If $p \in \partial^*B$, then $(\partial[p]_P)^* = (\partial[p]_B)^*$ is a ball, so $p \in Q$. Thus $B \cap Q = \partial^*B$. In other words, B^* is a ball with boundary $(\partial[q]_Q)^*$, as desired. \square

Lemma 3.50. *A poset P is a cell complex with coboundary Q if and only if the amalgam $P \cup_Q C^*Q$ is a cell complex.*

Proof. Let $q \in Q$ and write $P \div Q = P \cup_Q C^*Q$. Then $\partial[q]_{P+} \simeq \partial[q]_P \div \partial[q]_Q$. So it suffices to show that a poset B is a ball if and only if $B^+ := (B^* \div (\partial B)^*)^*$ is a sphere. We have $B^+ \simeq B \cup_{\partial B} C(\partial B)$. If B is a ball, then so is $C(\partial B)$, hence B^+ is a sphere. If B^+ is a sphere, then by Theorem 3.46 it is a cell complex. Then $C(\partial B)$ is a ball, hence B is a ball. \square

Corollary 3.51. *If $\alpha: K' \rightarrow K$ is a subdivision map and K is a cell complex with coboundary, then K' is a cell complex with coboundary $\alpha^{-1}(\partial^*K)$.*

3.F. Comanifolds

3.52. Cellular map. A monotone map of posets $f: P \rightarrow Q$ will be called *cellular*, if $f^{-1}(\lfloor q \rfloor)$ is a cell complex with coboundary $f^{-1}(\partial^*\lfloor q \rfloor)$ for each $q \in Q$.

Theorem 3.53. *Let $f: K \rightarrow L$ be a simplicial map of simplicial complexes. Then f is cellular.*

Proof. Let us fix a $\sigma \in L$, and let $D = f^{-1}(\lfloor \sigma \rfloor)$ and $E = f^{-1}(\partial^*\lfloor \sigma \rfloor)$. Pick some $\tau \in D$. Then $\lceil \tau \rceil = \lceil \mu \rceil * \lceil \nu \rceil$, where $f(\mu) = \sigma$ and $f(\nu) \cap \sigma = \emptyset$. We note that $\nu \neq \emptyset$ if and only if $\tau \in E$. The isomorphism $\lceil \tau \rceil_K \simeq \lceil \mu \rceil_K * \lceil \nu \rceil_K$ can be equivalently written as $C^*\lceil \tau \rceil_K \simeq C^*\lceil \mu \rceil_K \times C^*\lceil \nu \rceil_K$. The latter restricts to an isomorphism $\lceil \tau \rceil_D \simeq \lceil \mu \rceil_D \times C^*\lceil \nu \rceil_K$,

given by $\tau' \mapsto (\mu', \nu')$, where $\lceil \tau' \rceil = \lceil \mu' \rceil * \lceil \nu' \rceil$. By Lemma 2.42 $\lceil \mu \rceil_D = CR^*$, where R is a sphere (in fact, a join of boundaries of simplices).

If $\nu = \emptyset$, then $\lceil \tau \rceil_D = \lceil \mu \rceil_D = CR^*$, hence $(\partial \lceil \tau \rceil_D)^* = R$, which is a sphere. If $\nu \neq \emptyset$, then $\lceil \tau \rceil_D \simeq CR^* \times C^* \lceil \nu \rceil_K$. We may write $\lceil \nu \rceil_K = CQ^*$, where Q is a sphere (in fact, the boundary of a simplex). By the associativity of prejoin $C^*(CQ^*) \simeq C(C^*Q^*) \simeq C(CQ)^*$. Hence $\lceil \tau \rceil_D \simeq CR^* \times C(CQ)^*$. Then $\lceil \tau \rceil_D^* \simeq C^*R \times C^*(CQ) \simeq C^*(R * CQ)$. Therefore $(\partial \lceil \tau \rceil_D)^* \simeq R * CQ$, which is a ball. Thus by Lemma 3.49, D is a cell complex with coboundary E . \square

By Remark 3.44 if f is cellular, then $f^*: P^* \rightarrow Q^*$ is a stratification map. This yields the following corollary of Theorem 3.53, which was known in different terms [22; proof of Lemma 2.2] (see also [18; p. 35]):

Corollary 3.54. *If $f: K \rightarrow L$ is a simplicial map between simplicial complexes, then $f^*: K^* \rightarrow L^*$ is a stratification map.*

3.55. Comanifold. A monotone map of posets $f: P \rightarrow Q$, will be called a *[pseudo-]comanifold* if $f^{-1}(\lceil \sigma \rceil)$ is a [pseudo*-]manifold with boundary $f^{-1}(\partial \lceil \sigma \rceil)$ for each $\sigma \in Q$.

Depending on the structure of Q , a [pseudo-]comanifold $f: P \rightarrow Q$ may have dimension and/or codimension. If Q^* has pure cones, then f is of *dimension* d if $f^{-1}(\lceil \sigma \rceil)$ is of dimension $d - i$ for each $\sigma \in Q$ with $\lceil \sigma^* \rceil$ of dimension i .

More important is codimension. If Q itself has pure cones, then f of *codimension* k , or simply is a *k-[pseudo-]comanifold*, if $f^{-1}(\lceil \sigma \rceil)$ is of dimension $i - k$ for each $\sigma \in Q$ with $\lceil \sigma \rceil$ of dimension i .

When Q is a cell complex, k -comanifolds are a combinatorial form of “ k -mock bundles”, introduced by Buoncrisiano, Rourke and Sanderson; k -[pseudo-]comanifolds work as geometric representatives of k -dimensional unoriented PL cobordism [resp. mod 2 cohomology] classes of $|Q|$ [18].

When Q is purely n -dimensional, both dimension d and codimension k are defined for [pseudo-]manifolds into Q ; obviously, $k = n - d$.

If f is a pseudo-comanifold, then f^* is a cellular map, and in particular f itself is a stratification map. When P is a cell complex, any subdivision of P is a comanifold.

Corollary 3.56. *Let $f: M \rightarrow N$ be a simplicial map of simplicial complexes. If M is an m -[pseudo-]manifold, then $f^*: M^* \rightarrow N^*$ is a [pseudo-]comanifold of dimension m .*

The manifold part is essentially Cohen’s theorem [20; 5.6]; concerning the pseudo-manifold part see [34] and [25].

When N is purely n -dimensional, we can add to Corollary 3.56 that f^* is an $(n - m)$ -[pseudo-]comanifold; and if additionally N^* is a cell complex (which is equivalent to N being a manifold), then f^* represents an $(n - m)$ -dimensional cobordism [resp. cohomology] class $[f^*]$ of $|N|$. One more application of Corollary 3.56 ensures that $[f^*]$ depends only on the m -dimensional bordism [resp. homology] class $f_*([M])$. Thus Corollary 3.56

is really a combinatorial description of the Poincaré duality homomorphism. (A geometric proof that this homomorphism is an isomorphism involves a transversality argument, see [18].)

Proof. Let $\sigma \in N$. By Theorem 3.53, $D := f^{-1}(\lfloor \sigma \rfloor)$ is a cell complex with coboundary $E := f^{-1}(\partial^* \lfloor \sigma \rfloor)$. On the other hand, since M is a [pseudo-]manifold, M^* [resp. $(M^*)^{(1)}$] is a cell complex. Since $\lfloor \sigma^* \rfloor$ is closed in N^* , and $f^*: Y^* \rightarrow X^*$ is monotone, $D^* = (f^*)^{-1}(\lfloor \sigma^* \rfloor)$ is closed in M^* . Hence D^* [resp. $(D^*)^{(1)}$] is a cell complex. Thus D is a [pseudo-]manifold. The assertion on its dimension follows by the proofs of Lemma 2.42 and Theorem 3.53. \square

Remark 3.57. It is implicit in the above proof that a monotone map $f: P \rightarrow Q$ is a [pseudo-]comanifold if and only if P [resp. $P^{(1)}$] is a cell complex and $f^*: P^* \rightarrow Q^*$ is cellular.

Example 3.58. Let $\pi: P \times Q \rightarrow P$ be the projection. Then

- π and π^* are stratification maps;
- if P and Q are cell complexes, then π is cellular;
- if P and Q are manifolds, then π and π^* are comanifolds.

The assertions on π^* follow from the corresponding assertions on π since $\pi^*: P^* \times Q^* \rightarrow P^*$ is of the same form as π . We now prove the three assertions on π . The first assertion follows from Corollary 3.30, since identity and constant maps are obviously stratification maps. To prove the second [resp. third] assertion, it suffices to note that if K is a cell complex [resp. manifold] with coboundary ∂K , then clearly $K \times Q$ is a cell complex [resp. manifold] with coboundary $\partial K \times Q$.

Example 3.59. Let $\iota: Q \hookrightarrow P$ be an embedding onto a closed subposet. Then

- ι^* is a stratification map;
- if $Q^\#$ is a cell complex, then ι is cellular;
- if Q is a manifold, then ι^* is a comanifold.

We will identify Q with its image in P . The first assertion was proved in Lemma 3.31(a). If Q is additionally a cell complex, and $q \in \partial^* \lfloor p \rfloor_Q$, then $\partial \lfloor q \rfloor_{\lfloor p \rfloor_Q} = \partial \lfloor p, q \rfloor$. By Lemma 3.42 the latter is the dual cone over a sphere. Hence its dual is a ball, so $\lfloor p \rfloor_Q$ is a cell complex with coboundary $\partial^* \lfloor p \rfloor_Q$. Finally, if Q is a manifold, then $\partial^* \lfloor p \rfloor_Q$ is a sphere, so $\lfloor p^* \rfloor_{Q^*}$ is a ball (in particular, a manifold) with boundary $\partial \lfloor p^* \rfloor_{Q^*}$.

Corollary 3.56 shows how to get a comanifold from a manifold. Theorem 3.60(b) below is a converse result, producing a manifold from a comanifold (even in a slightly more general setup).

Theorem 3.60. (a) Let $f: P \rightarrow Q$ be a cellular map, and Z an open subposet of Q . If Q is a cell complex with coboundary Z , then P is a cell complex with coboundary $f^{-1}(Z)$; when f is surjective, the converse holds.

(b) Let $f: M \rightarrow N$ be a [pseudo-]comanifold of dimension m , where N^* is a cell complex with coboundary Z^* . Then M is an m -[pseudo*-]manifold with boundary $f^{-1}(Z)$.

In the case where N is a manifold with boundary, the manifold part of (c) is essentially the amalgamation lemma of Buoncrisiano–Rourke–Sanderson [18; Lemma II.1.2].

Proof. (a). By Remark 3.44, $f^*: P^* \rightarrow Q^*$ is a stratification map, and $f^{-1}(q)$ is a cell complex for each $q \in Q$. Let $p \in Y := f^{-1}(Z)$, and write $F = \partial[p]_{f^{-1}(f(p))}$. By Theorem 3.25 (applied to f^*), $(f^*)_p: (\partial[p]_P)^* \rightarrow F^* + (\partial[f(p)]_Q)^*$ is a subdivision map. Then by Theorem 3.4 there is a homeomorphism $|\partial[p]_P| \rightarrow |\partial[f(p)]_Q + F|$. By the hypothesis, F is a sphere. Hence by [52], $|\partial[p]_P|$ is a sphere [resp. a ball] if and only if $|\partial[f(p)]_Q|$ is a sphere [resp. a ball]. The assertion now follows from Lemma 3.49. \square

(b). This follows immediately from (b) and Remark 3.57. \square

Corollary 3.61. *Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be monotone maps of posets.*

(a) *If f and g are cellular, then so is gf . If f and gf are cellular, and f is surjective, then g is cellular.*

(b) *If f and g are comanifolds, then so is gf .*

3.62. Quasi-cellular maps. Let us call a monotone map $f: P \rightarrow Q$ *quasi-cellular* if f^* is a stratification map and $f^{-1}(q)$ is a cell complex for each $q \in Q$. By Remark 3.44, cellular maps are quasi-cellular.

Remark 3.63. We note that the proof of Theorem 3.60(a) works under the weaker hypothesis that f is quasi-cellular.

From Remark 3.63 (in the case $Z = \emptyset$) and Corollary 3.28 we obtain

Corollary 3.64. *Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be monotone maps of posets. If f and g are quasi-cellular, then so is gf . If f and gf are quasi-cellular, and f is surjective, then g is quasi-cellular.*

The following lemma should be compared with Remark 3.57.

Lemma 3.65. (a) *A monotone map $f: P \rightarrow Q$ is cellular if and only if f is quasi-cellular and $Q^\#$ is a cell complex.*

(b) *If $f: P \rightarrow Q$ is a cellular map, then $P^\#$ is a cell complex.*

Proof. (a). It follows from Lemma 3.42 that $P^\#$ is a cell complex if and only if $\text{lk}(q, Q)$ is a cell complex for each $q \in Q$. On the other hand, by Remark 3.44, f is cellular if and only if f is quasi-cellular and $f^{-1}(\text{lk}(q, Q))$ is a cell complex for each $q \in Q$. Thus it suffices to show the following: assume that f is quasi-cellular and $q \in Q$; then $\text{lk}(q, Q)$ is a cell complex if and only if $f^{-1}(\text{lk}(q, Q))$ is.

But if f is quasi-cellular, then so is its restriction to $f^{-1}(\text{lk}(q, Q))$. The assertion now follows from Remark 3.63. \square

(b). By Lemma 3.42 it suffices to show that every open interval $\partial^* \partial[p, q]$ is a sphere. Let $p, q \in P$ with $p < q$. Since f is cellular, $R := f^{-1}(\lfloor f(p) \rfloor)$ is a cell complex with coboundary. Then $S := (\partial[q]_R)^*$ is either a sphere or a ball; in any case, it is a cell complex. So $\partial[p^*]_S$ is a sphere. Its dual $(\partial[p^*]_S)^*$, which is also a sphere, is isomorphic to $\partial^* \partial[p]_{S^*} = \partial^* \partial[p, q]$. \square

3.G. Multi-collaring

Given a monotone map $f: P \rightarrow Q$, the simplicial map $f^\flat: P^\flat \rightarrow Q^\flat$ is dual to a monotone map of the barycentric handle decompositions $H_f: H(P) \rightarrow H(Q)$. On the other hand, the subdivision map $\flat: P^\flat \rightarrow P$ is dual to a monotone map $\bar{R}_P: H(P) \rightarrow P^*$. The composition $P^\flat \simeq (P^*)^\flat \xrightarrow{\flat} P^*$ is dual to a monotone map $R_P: H(P) \rightarrow P$.

Theorem 3.66. *Let $f: P \rightarrow Q$ be a stratification map. Then for each $q \in Q$ there exists a commutative diagram*

$$\begin{array}{ccc} H_f^{-1}(H_q^Q) & \xrightarrow{\alpha_q} & H_f^{-1}(H_q^{[q]}) \times H_q^{[q]} \\ \downarrow H_f| & & \downarrow (H_f|) \times \text{id} \\ H_q^Q & \xrightarrow{\simeq} & H_q^{[q]} \times H_q^{[q]}, \end{array}$$

where α_q is a fiberwise subdivision map with respect to the composition $H_f^{-1}(H_q^{[q]}) \times H_q^{[q]} \xrightarrow{\pi} H_q^{[q]} \xrightarrow{\bar{R}_Q|} [q]^*$.

Remark 3.67. Theorem 3.66 implies that a stratification map gives rise to a PL variety filtration in the sense of D. Stone [64], and in particular to a locally conelike TOP stratified set in the sense of Siebenmann [60]. As discussed in [64] and [60], these generalize Whitney stratified spaces and Thom's topologically stratified spaces; see also Grothendieck's reflections on "tame topology" and "dévissage" [35; §§5,6 and endnote (6)].

Proof. If $Q = [q]$ or more generally $[q]_Q = \{q\}$, we let $\alpha = \text{id}$. Arguing by induction, we may assume that the assertion is known for $Q' := Q \setminus \{r\}$ and $P' := f^{-1}(Q')$ and $f' := f|_{P'}$, for some maximal element r of Q . We may assume that $r > q$ (otherwise the inductive step is trivial).

Let $S = f^{-1}([r])$. Since $f|_S$ is a stratification map, $\partial S = f^{-1}(\partial[r])$. By the combinatorial invariance of link, $(\partial S)^\flat = \partial(S^\flat)$ so we may omit the brackets. Since ∂S^\flat is a collared full subcomplex of S^\flat , by the proof of Lemma 3.18, $[\partial S^\flat] \simeq MC(\beta^*)$, where $\beta^*: [\partial S^\flat] \setminus \partial S^\flat \rightarrow \partial S^\flat$ is dual to a subdivision map.

From the construction of β it is easy to see that $\beta^{-1}(H_f^{-1}(H_q^{\partial[r]})) = H_f^{-1}(H_q^{[r]} \setminus H_q^{\partial[r]})$. It follows that $H_f^{-1}(H_q^{[r]}) \simeq MC^*(\beta_q)$, where $\beta_q: H_f^{-1}(H_q^{[r]} \setminus H_q^{\partial[r]}) \rightarrow H_f^{-1}(H_q^{\partial[r]})$ is the restriction of β . Then Corollary 3.15 yields a subdivision map $\alpha_1^r: H_f^{-1}(H_q^{[r]}) \rightarrow H_f^{-1}(H_q^{\partial[r]}) \times [2]$ over $[2]$. In the case that $f = \text{id}$, this subdivision map in the domain specializes to an isomorphism in the range, $H_q^{[r]} \simeq H_q^{\partial[r]} \times [2]$, which is the identity on $H_q^{\partial[r]} = H_q^{\partial[r]} \times \{2\}$. Since $[q, r] = [r]_{[q]}$, the latter isomorphism in turn specializes to $H_q^{[q, r]} \simeq H_q^{\partial[q, r]} \times [2]$.

Since α_1^r is the identity on $H_f^{-1}(H_q^{\partial[r]})$, it extends by the identity on $H_f^{-1}(H_q^{Q'})$ to a subdivision map

$$\alpha_1: H_f^{-1}(H_q^Q) \rightarrow H_f^{-1}(H_q^{Q'}) \cup_{H_f^{-1}(H_q^{\partial[r]})} H_f^{-1}(H_q^{\partial[r]}) \times [2]$$

over $([q]_{Q'} \cup \partial_{[q,r]} \partial[q,r] \times [2])^*$. In more detail, the subdivision map is fiberwise with respect to the obvious map extending the composition

$$H_f^{-1}(H_q^{Q'}) \xrightarrow{H_f|} H_q^{Q'} \simeq H_q^{[q]} \times H_q^{[q]_{Q'}} \xrightarrow{\pi} H_q^{[q]_{Q'}} \xrightarrow{\bar{R}_{Q'}|} [q]_{Q'}^*.$$

On the other hand, the inductive hypothesis yields a subdivision map $\alpha'_2: H_f^{-1}(H_q^{Q'}) \rightarrow H_f^{-1}(H_q^{[q]}) \times H_q^{[q]_{Q'}}$ over $[q]_{Q'}^*$. By Lemma 3.9 the latter restricts to a subdivision map $\alpha_2^r: H_f^{-1}(H_q^{\partial[q,r]}) \rightarrow H_f^{-1}(H_q^{[q]}) \times H_q^{\partial[q,r]}$ over $(\partial[q,r])^*$. Then α'_2 and $\alpha_2^r \times \text{id}_{[2]}$ combine into a subdivision map

$$\alpha_2: H_f^{-1}(H_q^{Q'}) \bigcup_{H_f^{-1}(H_q^{\partial[q,r]})} H_f^{-1}(H_q^{\partial[q,r]}) \times [2] \rightarrow H_f^{-1}(H_q^{[q]}) \times \left(H_q^{[q]_{Q'}} \bigcup_{H_q^{\partial[q,r]}} H_q^{\partial[q,r]} \times [2] \right)$$

over $([q]_{Q'} \cup \partial_{[q,r]} \partial[q,r] \times [2])^*$. The range of α'_2 is isomorphic to $H_f^{-1}(H_q^{[q]}) \times H_q^{[q]_{Q'}}$ over the composition $([q]_{Q'} \cup \partial_{[q,r]} \partial[q,r] \times [2])^* \rightarrow ([q]_{Q'} \cup \partial_{[q,r]} [q,r])^* \simeq [q]_{Q'}^*$. Hence α_1 followed by α_2 is a subdivision map $H_f^{-1}(H_q^Q) \rightarrow H_f^{-1}(H_q^{[q]}) \times H_q^{[q]_{Q'}}$ over $[q]_{Q'}^*$, as desired. The commutativity of the diagram is clear from the construction. \square

Corollary 3.68. *If $f: P \rightarrow Q$ is a subdivision map, then so is $H_f: H(P) \rightarrow H(Q)$.*

Proof. Let $q \in Q$, and write $R = [q]$. We have homeomorphisms

$$(|H_f^{-1}(H_q^R)|, |H_f^{-1}(\partial H_q^R)|) \cong (|f^{-1}(R)|, |f^{-1}(\partial R)|) \cong (|R|, |\partial R|) \cong (|H_q^R|, |\partial H_q^R|),$$

the first and the third one being given by Lemma 3.18, and the second one by the hypothesis. The composition of these three homeomorphisms along with Theorem 3.66 yield a homeomorphism $(|H_f^{-1}(H_q^Q)|, |H_f^{-1}(\partial H_q^Q)|) \cong (|H_q^Q|, |\partial H_q^Q|)$, as desired. \square

3.H. Pullback

McCrory has shown that PL transversality of subpolyhedra is a symmetric relation [48]: subpolyhedra X and Y of a closed PL manifold M are PL transverse if and only if (M, X, Y) is PL homeomorphic to (P, Q, R) , where P is a poset, Q a closed subposet of P and R an open subposet of P .

On the other hand, Buoncrisiano, Rourke and Sanderson implicitly extended the notion of PL transversality to pairs of PL maps [18; “Induced mock bundles” (p. 23) and “Extension to polyhedra” (p. 35)]: PL maps $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ between polyhedra may be called PL transverse if f and g admit triangulation by simplicial maps $F: P \rightarrow K$ and $G: Q \rightarrow K'$ such that K' is an affine simplicial subdivision of K , and the composition $Q \xrightarrow{G} K' \xrightarrow{\alpha} K$, where α is the subdivision map, is a stratification map.

The following example shows that McCrory’s symmetric definition does not extend to maps, i.e. one cannot generalize the simplicial map F to an arbitrary monotone map whose dual is a stratification map.

Example 3.69. Let $\alpha: (\partial\Delta^1) * (\partial\Delta^2) \rightarrow \partial\Delta^3$ be the stellar subdivision map of a facet F of the boundary of the tetrahedron. Up to isomorphism, α^* is the same as the quotient

map $f: \partial(\Delta^1 \times \Delta^2) \rightarrow \partial\Delta^3$, shrinking $\{a\} \times \Delta^2$ to a vertex p of Δ^3 , for some vertex a of Δ^1 .

Now let us identify the range of f with the range of α (rather than the dual of the range of α as we did before) so that p is identified with a vertex of F . Then we may consider the pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \partial(\Delta^1 \times \Delta^2) \\ g \downarrow & & f \downarrow \\ (\partial\Delta^1) * (\partial\Delta^2) & \xrightarrow{\alpha} & \Delta^3. \end{array}$$

Then β is not a subdivision map, and not even a stratification map; by symmetry, it follows then that g^* is also not a stratification map.

Indeed, let us consider the restriction of the above diagram over the 2-simplex F . Up to isomorphism, it is the same as the following pullback diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{\beta_0} & \Delta^1 \times \Delta^1 \\ g_0 \downarrow & & f_0 \downarrow \\ \Delta^0 * \partial\Delta^2 & \xrightarrow{\alpha_0} & \Delta^2. \end{array}$$

Here α_0 is the stellar subdivision map of the 2-simplex, and f_0 is the quotient map, shrinking one side of the quadrilateral $\Delta^1 \times \Delta^1$ onto the vertex p of the triangle Δ^2 .

Let $q \in \Delta^2$ be the maximal element. Then $f_0^{-1}(p) \simeq \Delta^1$ and $\alpha_0^{-1}(q) \simeq (\Delta^2)^*$. On the other hand, $f_0^{-1}(q)$ is a singleton $\{q'\}$ and $\alpha_0^{-1}(p)$ is a singleton $\{p'\}$. Let $Q = [p'] \cap \alpha_0^{-1}(q)$; thus $Q \simeq (\Delta^1)^*$.

For each pair $(\bar{p}, \bar{q}) \in f_0^{-1}(p) \times \alpha_0^{-1}(q)$ we have $f_0(\bar{p}) = p = \alpha_0(p')$ and $f_0(q') = q = \alpha_0(\bar{q})$. Hence $(\bar{p}, p') \in X_0$ and $(q', \bar{q}) \in X_0$. Moreover, $\bar{p} < q'$ in $\Delta^1 \times \Delta^1$, and if $\bar{q} \in Q$, then $p' < \bar{q}$ in Δ^2 . Hence $(\bar{p}, p') < (q', \bar{q})$ in X as long as $\bar{q} \in Q$. Therefore X_0 contains a copy of the prejoin $\Delta^1 + (\Delta^1)^*$. Then $|X_0|$ is 3-dimensional, and so β_0 cannot be a subdivision map by Theorem 3.4. Also β_0 is not a stratification map, since Δ^1 is not collared in $\Delta^1 + (\Delta^1)^*$. Then also β cannot be a stratification map since β_0 is its restriction over a closed subposet.

The remainder of this section is concerned with finding ways to avoid the above example.

Lemma 3.70. *Consider a pullback diagram*

$$\begin{array}{ccc} P' & \xrightarrow{\beta} & P \\ f' \downarrow & & f \downarrow \\ Q' & \xrightarrow{\alpha} & Q. \end{array}$$

If f is cubical, then so is f' . If additionally α is a subdivision map or a stratification map, then so is β .

Proof. Given a $p' = (q', p) \in P'$, we have $q' \in \alpha^{-1}(f(p))$, and consequently $[p'] \subset \alpha^{-1}(\lceil f(p) \rceil) \times [p]$. Hence in order to prove that f' is cubical, we may assume without loss of generality that $P = [p]$ and $Q = \lceil f(p) \rceil$. Clearly, we may also assume this without loss of generality in proving the assertion on β .

Then f is isomorphic to the projection $Q \times CR \rightarrow Q$ for some poset R . Hence f' is isomorphic to the projection $Q' \times CR \rightarrow Q'$, and therefore is cubical. On the other hand, β is isomorphic to $\alpha \times \text{id}_{CR}: Q' \times CR \rightarrow Q \times CR$. If α is a stratification map, then $\partial Q' = \alpha^{-1}(\partial Q)$. The assertion on β now follows from Corollary 3.30. \square

Corollary 3.71. *Consider the following diagram consisting of three pullback squares:*

$$\begin{array}{ccccc}
 Z & \xrightarrow{h} & Y & \xrightarrow{\bar{\rho}_Y} & R \\
 \chi \downarrow & & \psi \downarrow & & \varphi \downarrow \\
 X & \xrightarrow{g} & h(Q) & \xrightarrow{\bar{r}_Q} & Q^* \\
 \rho_X \downarrow & & r_Q \downarrow & & \\
 P & \xrightarrow{f} & Q & &
 \end{array}$$

Then r_Q , \bar{r}_Q , ρ_X and ρ_Y as well as both vertical compositions and both horizontal compositions in the diagram are cubical. If f is a stratification map or a subdivision map, then so are g and h , and if φ is a stratification map or a subdivision map, then so are ψ and χ .

Proof. Let us pick a cone $A := \lceil [\sigma, \tau]^* \rceil$ of $h(Q)$. The restrictions $s: A \rightarrow \lceil \sigma \rceil$ and $t: A \rightarrow \lceil \tau^* \rceil$ of r_Q and \bar{r}_Q are isomorphic to the projections of $\lceil \sigma \rceil \times \lceil \tau^* \rceil$ onto the two factors. Hence r_Q and \bar{r}_Q are cubical.

Let $P_0 = f^{-1}(\lceil \sigma \rceil)$ and $X_0 = g^{-1}(A)$, and consider the restriction $f_0: P_0 \rightarrow \lceil \sigma \rceil$ of f . Then the restriction $g_0: X_0 \rightarrow A$ of g is isomorphic to $f_0 \times \text{id}_{\lceil \tau^* \rceil}: P_0 \times \lceil \tau^* \rceil \rightarrow \lceil \sigma \rceil \times \lceil \tau^* \rceil$. Hence the composition $X_0 \xrightarrow{g_0} A \xrightarrow{t} \lceil \tau^* \rceil$ is isomorphic to the projection $P_0 \times \lceil \tau^* \rceil \rightarrow \lceil \tau^* \rceil$. Thus the composition $X \xrightarrow{g} h(Q) \xrightarrow{\bar{r}_Q} Q^*$ is cubical. By symmetry, the composition $Y \xrightarrow{\psi} h(Q) \xrightarrow{r_Q} Q$ is also cubical.

The remaining assertions now follow using Lemma 3.70. \square

Theorem 3.72. *Consider a pullback diagram*

$$\begin{array}{ccc}
 X & \xrightarrow{\beta} & h(Q) \\
 f \downarrow & & r_Q \downarrow \\
 P & \xrightarrow{\alpha} & Q.
 \end{array}$$

If α is a stratification map, then $f^: X^* \rightarrow P^*$ is a fiberwise subdivision map with respect to $\alpha^*: P^* \rightarrow Q^*$ and the composition $X^* \xrightarrow{\beta^*} Q^\# \xrightarrow{\#} Q$.*

Beware that the range of the composition $\# \beta^*$ is Q (and not Q^*), so f^* is the peculiar kind of a fiberwise subdivision map.

Proof. Let $p \in P$. It suffices to show that $f^*| : (f^*)^{-1}([p]^*) \rightarrow [p]^*$ is a fiberwise subdivision map with respect to the restrictions of α^* and $\# \beta^*$. By Lemma 3.31(a) the inclusion $[p] \hookrightarrow P$ is a stratification map, hence by Corollary 3.28, $\alpha|_{[p]}$ is a stratification map. Thus we may assume that $P = [p]$. By Lemma 3.31(b) without loss of generality α is surjective, in particular, $Q = [\alpha(p)]$.

Let r be a maximal element of Q . Let $R = \alpha^{-1}([r])$. Since α is a stratification map, $\partial R = \alpha^{-1}(\partial[r])$. Let $Q' = Q \setminus \{r\}$ and $P' = \alpha^{-1}(Q')$. Since α is surjective, $P' \neq P$. Let $X' = f^{-1}(P')$ and $X'_0 = f^{-1}(\partial^* P')$. Similarly, let $X = f^{-1}(P)$ and $X_0 = f^{-1}(\partial^* P)$. Arguing by induction, we may assume that $f^*|_{(X')^*}$ is a fiberwise subdivision map with respect to $\alpha^*|_{(P')^*}$ and $\# \beta^*|_{(X')^*}$. Then, in particular, there exists a homeomorphism $\varphi : |X'| \rightarrow C|X'_0|$ keeping $|X'_0|$ fixed and sending $|Y_q|$ onto $C|X'_0 \cap Y_q|$ for each $q \in Q'$, where $Y_q = \beta^{-1}(h([q]))$. In particular, φ restricts to a homeomorphism $\varphi_0 : |Z_r| \rightarrow C|X'_0 \cap Z_r|$ keeping $|X'_0 \cap Z_r|$ fixed, where $Z_r = \beta^{-1}(h(\partial[r]))$. In order to show that φ extends to a homeomorphism $\varphi^+ : |X| \rightarrow C|X_0|$ keeping $|X_0|$ fixed and sending $|Y_q|$ onto $C|X_0 \cap Y_q|$ for each $q \in Q$, it suffices to show that φ_0 extends to a homeomorphism $\varphi_0^+ : |Y_r| \rightarrow C|X_0 \cap Y_r|$ keeping $|X_0 \cap Y_r|$ fixed.

By Lemma 2.80(b), $h([r]) \simeq MC(r_{\partial[r]}) \cup_{\partial[r]} [r]$. Therefore by Lemma 2.65, $Y_r \simeq MC(f|_{Z_r}) \cup_{\partial R} R$ keeping Z_r fixed; by the same token, this isomorphism sends $X_0 \cap Y_r$ onto $MC(f|_{X_0 \cap Z_r}) \cup_{\partial^* \partial R} \partial^* R$. By Lemma 3.9, $f^*|_{Z_r^*} : Z_r^* \rightarrow (\partial R)^*$ is a subdivision map. Then by Corollary 3.15 there exists a monotone map $\gamma : MC(f|_{Z_r}) \rightarrow (\partial R) \times [2]$ such that γ^* is a subdivision map over $[2]$. Hence by Theorem 3.4, $|MC(f|_{Z_r})|$ is homeomorphic to $|(\partial R) \times [2]|$ “over $[2]$ ”. Since $(\partial^* \partial R)^*$ is closed in $(\partial R)^*$, it follows that this homeomorphism sends $|MC(f|_{X_0 \cap Z_r})|$ onto $|(\partial^* \partial R) \times [2]|$. Thus $(|Y_r|, |Z_r|) \cong (|R \cup_{\partial R} (\partial R) \times I|, |\partial R \times \{1\}|)$ by a homeomorphism sending $(|X_0 \cap Y_r|, |X_0 \cap Z_r|)$ onto $(|\partial^* R \cup_{\partial^* \partial R} (\partial^* \partial R) \times I|, |\partial^* \partial R \times \{1\}|)$.

On the other hand, since α is a stratification map, ∂R is collared in R . Since $\partial^* R$ is open in R , by Theorem 3.19 and Addendum 3.20, $(|R|, |\partial R|) \cong (|R \cup_{\partial R} (\partial R) \times I|, |\partial R \times \{1\}|)$ by a homeomorphism sending $(|\partial^* R|, |\partial^* \partial R|)$ onto $(|\partial^* R \cup_{\partial^* \partial R} (\partial^* \partial R) \times I|, |\partial^* \partial R \times \{1\}|)$. Thus $(|Y_r|, |Z_r|) \cong (|R|, |\partial R|)$ by a homeomorphism sending $(|X_0 \cap Y_r|, |X_0 \cap Z_r|)$ onto $(|\partial^* R|, |\partial^* \partial R|)$. Since $(|R|, |\partial R|)$ is homeomorphic to the cone over $(|\partial^* R|, |\partial^* \partial R|)$, this yields the desired extension φ_0^+ of φ_0 . \square

Theorem 3.73. *Consider the following diagram consisting of three pullback squares:*

$$\begin{array}{ccccc}
 Z & \xrightarrow{h} & Y & \xrightarrow{\bar{\rho}_Y} & R \\
 \chi \downarrow & & \psi \downarrow & & \varphi \downarrow \\
 X & \xrightarrow{g} & h(Q) & \xrightarrow{\bar{r}_Q} & Q^* \\
 \rho_X \downarrow & & r_Q \downarrow & & \\
 P & \xrightarrow{f} & Q & &
 \end{array}$$

If f is a subdivision map, then the composition $g_+ : X \xrightarrow{g} h(Q) \xrightarrow{\bar{r}_Q} Q^*$ is dual to a subdivision map, and if additionally φ is a stratification map, then the composition $h_+ : Z \xrightarrow{h} Y \xrightarrow{\bar{\rho}_Y} R$ is dual to a subdivision map.

Dually, if φ is a subdivision map, then the composition $\psi_+ : Y \xrightarrow{\psi} h(Q) \xrightarrow{r_Q} Q$ is dual to a subdivision map, and if additionally f is a stratification map, then the composition $\chi_+ : Z \xrightarrow{\chi} X \xrightarrow{\rho_X} P$ is dual to a subdivision map.

Proof. By symmetry it suffices to prove the assertions on g_+ and χ_+ .

By Theorem 3.72 and Lemma 3.10 (with $B = Q^*$) there exists a homeomorphism $|X^*| \rightarrow |P^*|$ sending $|(g_+^*)^{-1}(\lceil q \rceil)|$ onto $|(f^{-1}(\lceil q \rceil))^*|$ for each $q \in Q$. Since f is a subdivision map, we get that so is g_+^* .

It remains to show that χ_+^* is a subdivision map, assuming that φ is a subdivision map and f is a stratification map. Let $p \in P$. By Lemma 3.31(a) the inclusion $\lceil p \rceil \hookrightarrow P$ is a stratification map, hence by Corollary 3.28, $f|_{\lceil p \rceil}$ is a stratification map. Thus we may assume that $P = \lceil p \rceil$. Then by Lemma 3.31(b) without loss of generality $Q = \lfloor f(p) \rfloor$.

By Theorem 3.71 ρ_X is cubical. In particular, by Lemma 3.26 ρ_X^* is a stratification map. Hence $\partial^* X = \rho_X^{-1}(\partial^* P)$. By Corollary 3.71, χ is a subdivision map. We will now examine the proof of this assertion in order to show that χ is a fiberwise subdivision map with respect to the map $c : X \rightarrow [2]$ defined by $c^{-1}(2) = \partial^* X$.

Given an $x \in X$, let $q^* = g_+(x)$ and $p' = \rho_X(x)$. Then $g(x) = [f(p'), q]$. For convenience of reference we will mark the assertions. The restriction of r_Q [resp. \bar{r}_Q] to $\lceil [f(p'), q] \rceil$ is isomorphic to the projection of $\lceil f(p') \rceil \times \lceil q^* \rceil$ onto the first factor (1) [resp. onto the second factor (2)]. By (1), the restriction $\rho_x : \lceil x \rceil \rightarrow \lceil p' \rceil$ of ρ_X is isomorphic to the projection $\lceil p' \rceil \times \lceil q^* \rceil \rightarrow \lceil p' \rceil$ (3), and the restriction $g_x : \lceil x \rceil \rightarrow \lceil g(x) \rceil$ of g is isomorphic to the map $\lceil p' \rceil \times \lceil q^* \rceil \xrightarrow{f \times \text{id}} \lceil f(p') \rceil \times \lceil q^* \rceil$ (4). By (2) and (4), the restriction $g_x^+ : \lceil x \rceil \rightarrow \lceil g_+(x) \rceil$ of g_+ is isomorphic to the projection $\lceil p' \rceil \times \lceil q^* \rceil \rightarrow \lceil q^* \rceil$. Hence the restriction $\chi_x : \chi^{-1}(\lceil x \rceil) \rightarrow \lceil x \rceil$ of χ is isomorphic to the map $\lceil p' \rceil \times \varphi^{-1}(\lceil q^* \rceil) \xrightarrow{\text{id} \times \varphi|} \lceil p' \rceil \times \lceil q^* \rceil$ (5).

If $p' \neq p$, then $x \in \partial^* X$, and it is clear that the isomorphism in (3) sends the restriction $\partial^* \rho_x : \lceil x \rceil_{\partial^* X} \rightarrow \lceil p' \rceil_{\partial^* P}$ of ρ_x onto the projection $(\partial^* \lceil p' \rceil) \times \lceil q^* \rceil \rightarrow \partial^* \lceil p' \rceil$. It follows that the isomorphism in (5) sends the restriction $\partial^* \chi_x : \chi^{-1}(\lceil x \rceil_{\partial^* X}) \rightarrow \lceil x \rceil_{\partial^* X}$ of χ onto the map $(\partial^* \lceil p' \rceil) \times \varphi^{-1}(\lceil q^* \rceil) \xrightarrow{\text{id} \times \varphi|} (\partial^* \lceil p' \rceil) \times \lceil q^* \rceil$ (7). Since φ is a subdivision map, we have a homeomorphism $h : |\varphi^{-1}(\lceil q^* \rceil)| \rightarrow |\lceil q^* \rceil|$ sending $|\varphi^{-1}(\partial \lceil q^* \rceil)|$ onto $|\partial \lceil q^* \rceil|$. Then, in view of (5), $\text{id}_{|\lceil p' \rceil|} \times h$ is a homeomorphism between $|\chi^{-1}(\lceil x \rceil)|$ and $|\lceil x \rceil|$, which sends $|\chi^{-1}(\partial \lceil x \rceil)|$ onto $|\partial \lceil x \rceil|$ and, in view of (7), $|\chi^{-1}(\lceil x \rceil_{\partial^* X})|$ onto $|\lceil x \rceil_{\partial^* X}|$.

Thus χ is a fiberwise subdivision map with respect to the map $c : X \rightarrow [2]$, where $c^{-1}(2) = \partial^* X$. Hence by Lemma 3.10 there exists a homeomorphism $|Z| \cong |X|$ sending $|\chi^{-1}(\partial^* X)|$ onto $|\partial^* X|$. Since ρ_X^* is a subdivision map by Theorem 3.72, $|X|$ is homeomorphic to $C|\partial^* X|$ keeping $|\partial^* X|$ fixed. Hence $|Z|$ is homeomorphic to $C|\chi_+^{-1}(\partial^* \lceil p \rceil)|$ keeping $|\chi_+^{-1}(\partial^* \lceil p \rceil)|$ fixed. \square

4. CONSTRUCTIBLE POSETS AND MAPS

4.A. Constructibility

4.1. Constructible poset. We call a poset P [transversely] *constructible* if either P is a cone (i.e. has a greatest element) or $P = Q \cup R$, where Q and R are closed subposets of P such that $Q \cap R$ is of codimension one [resp. collared] both in Q and in R , and each of the three posets Q , R and $Q \cap R$ is [transversely] constructible. Such a decomposition $P = Q \cup R$ will be called a [transverse] *construction step* for brevity. Note that \emptyset is not constructible.

Remark 4.2. In general topology, a subset of a space X is called regular closed in X if it is equal to the closure of its interior. A subposet Q of a poset P is regular closed in P (with respect to the Alexandroff topology) iff $Q = \lceil S \rceil$, where S is a set of maximal elements of P ; and nowhere dense in P if it contains no maximal element of P .

If $P = Q \cup R$ is a construction step, it is easy to see that Q and R are regular closed in P , and $P \cap Q$ is nowhere dense in P and equals the frontier of Q as well as the frontier of R .

Lemma 4.3. *If $P = Q \cup R$ is a construction step, then $Q \cap R$ is of pure codimension one in Q and in R .*

Proof. It suffices to show that if P is a constructible poset, then an element of P that is covered by a maximal element is not covered by a non-maximal element.

This is clearly so if P is a cone. Suppose that $P = Q \cup R$ is a construction step, where Q and R satisfy the property in question. Suppose that a $p \in P$ is covered by a maximal element q of P and by a non-maximal element r of P . If $q, r \in Q$ or $q, r \in R$, this would contradict our hypothesis since Q and R are closed in P . Hence we may assume by symmetry that $q \in Q$ and $r \in R$. Then q is maximal in Q , r is non-maximal in R , and $p \in Q \cap R$.

If p is non-maximal in $Q \cap R$, then $p < q'$ for some maximal element q' of $Q \cap R$, which is covered by a maximal element of Q since $Q \cap R$ is of codimension one in Q . In particular, q' is non-maximal in Q , and then any $q'' \leq q'$ that covers p is non-maximal in Q . Since p is also covered by q , this contradicts our assumption that Q satisfies the property in question.

If p is maximal in $Q \cap R$, then p is covered by a maximal element r' of R since $Q \cap R$ is of codimension one in R . Since p is also covered by r , this contradicts our assumption that R satisfies the property in question. \square

Lemma 4.4. *Let P be a poset with pure cones. Then C^*P is constructible if and only if either P is a cone or the empty set, or $P = Q \cup R$, where Q and R are closed subposets of P such that $\dim |Q| = \dim |R| = \dim |Q \cap R| + 1$, and each of C^*Q , C^*R and $C^*(Q \cap R)$ is constructible.*

This implies that if K is an affine polytopal complex, then C^*K is constructible if and only if K is “constructible” in the sense of Hochster [40; p. 328], [37]. Note that C^*K is

precisely the “face poset” of K in sense of Topological Combinatorics (where the empty set is regarded a face).

Proof. If C^*P is constructible, then it is easy to see by induction that P is pure (using that C^*P has pure cones for the induction base). If P satisfies the property in question, then again P is pure by a similar reasoning. Now if P is pure, then the condition $\dim |Q| = \dim |R| = \dim |Q \cap R| + 1$ is equivalent to saying that $Q \cap R$ is of codimension one in Q and in R . \square

Lemma 4.5. *Let P be a poset.*

- (a) *If P is constructible, then $|P|$ is contractible.*
- (b) *If C^*P is constructible and P has pure cones, then $|P|$ is acyclic in dimensions $< \dim |P|$.*
- (c) *If P is [transversely] constructible, then so is C^*P ; the converse holds if P has pure cones and $|P|$ is acyclic.*

Part (b) is well-known in a special case (see [37; 2.22]).

Proof. The first assertion of (c) is obvious. Part (a) follows from the Mayer–Vietoris sequence and the Seifert–van Kampen theorem.

Next suppose that P has pure cones and C^*P is constructible, and let $n = \dim |P|$ and let r be the number of the intermediate stages C^*P_i in the construction process such that $P_i = \emptyset$. Then it is easy to see from the Mayer–Vietoris exact sequence that $\tilde{H}_k(|P|) = 0$ for $i < n$ and $\tilde{H}_n(|P|) = \mathbb{Z}^k$ (integer coefficients). This implies (b) and the second assertion of (c). \square

Lemma 4.6. (a) *If P is a constructible pseudo-manifold, then $|P|$ is a PL ball.*

(b) *If P is a transversely constructible cell complex, then $|P|$ is a PL ball.*

Part (a) is well-known in a special case (see [37; 2.16]).

Proof. Let P be a cell complex with coboundary and suppose that $P = Q \cup R$ is either a transverse construction step or a construction step where P is a pseudo-manifold. Then every maximal element of $Q \cap R$ is covered by precisely one element of Q , which is maximal in Q , and by precisely one element of R , which is maximal in R . Arguing by induction, we may assume that $|Q|$, $|R|$ and $|Q \cap R|$ are PL balls. Thus $|Q|$ and $|R|$ are balls of the same dimension n , and $|Q \cap R|$ is an $(n - 1)$ -ball lying in their boundaries. Then $|Q \cup R|$ is a ball by basic PL topology [67; Corollary to Theorem 2]. \square

Example 4.7. Let M be a triangulation of a contractible 4-manifold that is distinct from the 4-ball but has a 2-dimensional spine that 3-deforms to a point (e.g. Mazur’s manifold works). Then $M \times I$ is a non-constructible cellulation of the 5-ball.

Indeed, if $M \times I$ were constructible, then each construction step would have to be of the form $P \times I = Q \times I \cup R \times I$. Then M itself would be constructible, contradicting Lemma 4.6.

The non-transverse part of the following lemma is essentially known (in a special case), see [37; 2.14].

Lemma 4.8. *If P is a [transversely] constructible poset, then $\lfloor p \rfloor$ is [transversely] constructible for each $p \in P$.*

Proof. If P is a cone, then so is $\lfloor p \rfloor$. Assume that the assertion holds if P is [transversely] constructible in at most n steps. Suppose that P is [transversely] constructible in $n + 1$ steps. Then there is a [transverse] construction step $P = Q \cup R$, where Q , R and $Q \cap R$ are [transversely] constructible in at most n steps. If $p \notin Q$, then $\lfloor p \rfloor_P = \lfloor p \rfloor_R$ and the assertion follows by the inductive hypothesis. The case $p \notin R$ is similar. Note that the inclusion $\lfloor p \rfloor \subset P$ is a stratification map by Lemma 3.31(a). If $p \in Q \cap R$, then by Theorem 3.34 [resp. Lemma 3.27] $\lfloor p \rfloor_{Q \cap R}$ is of codimension one [resp. collared] in $\lfloor p \rfloor_Q$ and in $\lfloor p \rfloor_R$, and all three are [transversely] constructible by the inductive hypothesis. \square

4.9. Constructible map. A monotone map of posets $f: P \rightarrow Q$ will be called [transversely] constructible if it is a filtration [resp. stratification] map, and $f^{-1}(\lceil q \rceil)$ is [transversely] constructible for each $q \in Q$.

Lemma 4.10. *Let $f: P \rightarrow Q$ a [transversely] constructible map of posets, where Q is [transversely] constructible. Then P is [transversely] constructible.*

Proof. Let $Q = R \cup S$ be a [transverse] construction step. Then the restrictions of f to $f^{-1}(R)$, to $f^{-1}(S)$ and to $f^{-1}(R \cap S)$ are [transversely] constructible maps. Arguing by induction, we may then assume that $f^{-1}(R)$, $f^{-1}(S)$ and $f^{-1}(R \cap S)$ are [transversely] constructible. Since $R \cap S$ is of codimension one [resp. collared] in R and in S , the monotone map $g: Q \rightarrow I^*$ defined by $g(q) = \{0\}^*$ for $q \in R \setminus S$, $g(q) = \{1\}^*$ for $q \in S \setminus R$ and $g(q) = \{0, 1\}^*$ for $q \in R \cap S$ is a filtration [resp. stratification] map. By Corollary 3.28 [resp. 3.35], the composition $P \xrightarrow{f} Q \xrightarrow{g} I^*$ is a filtration [resp. stratification] map. Hence $f^{-1}(R \cap S)$ is of codimension one [resp. collared] in $f^{-1}(R)$ and in $f^{-1}(S)$. Thus $P = f^{-1}(R) \cup f^{-1}(S)$ is a [transverse] construction step. \square

Lemma 4.10 along with Corollary 3.28 [resp. 3.38] imply:

Theorem 4.11. *Composition of [transversely] constructible maps is a [transversely] constructible map.*

Lemma 4.12. *Let K be a poset and L a full subposet of K such that L^* is dense in K^* .*

- (a) *If L is closed in K , and K^* is [transversely] constructible, then so is L^* .*
- (b) *If L^* is constructible, then so is K^* .*

That L^* is dense in K^* is of course understood with respect to the Alexandroff topology, i.e., it is equivalent to $\lceil L^* \rceil = K^*$.

Proof. (a). Clearly, L^* is a cone if and only if so is K^* .

Suppose that $K^* = Q \cup R$ is a [transverse] construction step. Since L^* is open in K^* , the inclusion $\lfloor p \rfloor \subset P$ is a stratification map by Lemma 3.31(a). Hence by Theorem 3.27

[resp. Lemma 3.34] $L^* \cap Q \cap R$ is of codimension ≥ 1 [resp. collared] in $L^* \cap Q$ and in $L^* \cap R$. Since L is a closed subposet of K , its intersection with any subposet of K is closed in that subposet. Since L is a full subposet of K , its intersection with any open subposet of K is full in that subposet by Lemma 2.22(b).

Since maximal elements of Q and R are maximal in K^* , they lie in L^* . Hence $L^* \cap Q$ is dense in Q , and $L^* \cap R$ is dense in R . If p is a maximal element of $Q \cap R$, then since $Q \cap R$ is of codimension ≥ 1 in Q and in R , p is covered by a maximal element q of Q and by a maximal element r of R . Then $q, r \in L^*$, and since L is full in K , we get that $p \in L^*$. Therefore maximal elements of $Q \cap R$ lie in L^* . Hence $L^* \cap Q \cap R$ is dense in $Q \cap R$. Thus arguing by induction, we may assume that $L^* \cap Q$, $L^* \cap R$ and $L^* \cap Q \cap R$ are [transversely] constructible. \square

(b). Suppose that $L^* = Q \cup R$ is a construction step. Clearly $K^* = \lceil Q \cup R \rceil$ is the union of $\lceil Q \rceil$ and $\lceil R \rceil$. Maximal elements of Q , R , or $Q \cap R$ coincide with those of $\lceil Q \rceil$, $\lceil R \rceil$, or $\lceil Q \cap R \rceil$, respectively. Hence $\lceil Q \cap R \rceil$ is of codimension one in $\lceil Q \rceil$ and in $\lceil R \rceil$.

To prove that $\lceil Q \rceil \cap \lceil R \rceil$ is of codimension one in $\lceil Q \rceil$ and in $\lceil R \rceil$ it suffices to show that $\lceil Q \cap R \rceil = \lceil Q \rceil \cap \lceil R \rceil$. The inclusion $\lceil Q \cap R \rceil \subset \lceil Q \rceil \cap \lceil R \rceil$ is trivial. Given a $p \in \lceil Q \rceil \cap \lceil R \rceil$, we have $p \leq q$ and $p \leq r$ for some $q \in Q$ and some $r \in R$. Since L is full in K , the dual cone $\lfloor p \rfloor$ meets L^* in a dual cone $\lfloor s \rfloor$. Since $q, r \in L^*$, we have found an $s \in L^*$ such that $s \leq q$, $s \leq r$ and $p \leq s$. Hence $s \in Q \cap R$ and therefore $p \in \lceil Q \cap R \rceil$.

Arguing by induction, we would be able to assume that $\lceil Q \rceil$, $\lceil R \rceil$ and $\lceil Q \rceil \cap \lceil R \rceil$ are constructible once we show that Q^* , R^* and $Q^* \cap R^*$ are full respectively in $\lceil Q \rceil^*$, $\lceil R \rceil^*$ and $\lceil Q \rceil^* \cap \lceil R \rceil^*$. This follows from Lemma 2.22(b), since the former ones are the respective intersections of L with the latter ones. \square

Theorem 4.13. *If $f: K \rightarrow L$ is a simplicial map between simplicial complexes, or more generally a full map between posets such that f^* is a filtration map, then $f^*: K^* \rightarrow L^*$ is constructible if and only if $f^{-1}(\sigma)^*$ is constructible for each $\sigma \in L$.*

That the first hypothesis is a specialization of the second follows from Theorem 3.53, Remark 3.44 and Lemma 2.34.

Proof. By Lemma 2.22(b), $f^{-1}(\sigma)$ is full in $f^{-1}(\lfloor \sigma \rfloor)$ for each $\sigma \in L$. Also $f^{-1}(\sigma)$ is closed in $f^{-1}(\lfloor \sigma \rfloor)$ since $\{\sigma\}$ is closed in $\lfloor \sigma \rfloor$. Then by Lemma 4.12, $f^{-1}(\sigma)^*$ is constructible if and only if $\lceil f^{-1}(\sigma)^* \rceil_{K^*}$ is constructible. \square

Theorem 4.14. *If M is a poset such that $|M|$ is a closed PL manifold, and $f: M \rightarrow K$ is a constructible map such that f^* is cellular, then f is a subdivision map.*

This is a variation of the Cohen–Hommam theorem (Theorem 4.52 below).

Proof. By Theorem 3.46, M is a cell complex. Let $\sigma \in K$. Since $M_\sigma := f^{-1}(\lceil \sigma \rceil)$ is closed in M , it is a cell complex. Since f is a stratification, $\partial M_\sigma = f^{-1}(\partial \lceil \sigma \rceil)$. Since f^* is cellular, M_σ^* is a cell complex with coboundary $(\partial M_\sigma)^*$. Hence M_σ is a manifold with boundary ∂M_σ . Since M_σ is constructible, by Lemma 4.6(a) it is a ball. In particular, $|M_\sigma|$ is homeomorphic to $|C(\partial M_\sigma)|$ keeping $|\partial M_\sigma|$ fixed. \square

4.B. Zipping

4.15. Zipping. Let I denote the 1-simplex. Given a poset P and a $p \in P$, suppose that $\lceil p \rceil$ is isomorphic to $Q + I$ for some Q by an isomorphism h such that $\{p, q, r\} := h^{-1}(I)$ is full in P (in other words, p is the least upper bound of q and r in P). In this situation we say that P *elementarily zips* onto the quotient poset $P/\{p, q, r\}$ (see 2.60) *along* p . If additionally $\lceil p^* \rceil$ is collared in $\lceil q^* \rceil$ and in $\lceil r^* \rceil$, then we say that P *elementarily transversely zips* onto $P/h^{-1}(I)$ *along* p . A *[transverse] zipping* is a sequence of elementary [transverse] zippings.

Non-transverse zipping was introduced by N. Reading [56]; the author has been aware of the notion for a few years before he learned of Reading's paper from E. Nevo.

We recall that we have called a poset P *nonsingular* if it contains no interval of cardinality three.

Lemma 4.16. *If a nonsingular poset P zips onto Q , then*

- (a) *Q is nonsingular, and*
- (b) *the quotient map $f^*: P^* \rightarrow Q^*$ is constructible.*

Proof. (a). We may assume without loss of generality that there is an elementary zipping $f: P \rightarrow Q$ along a $p \in P$. Suppose that $[a, c] = \{a, b, c\}$ for some $a, c \in Q$. If $x \in Q$, $x \neq f(p)$, let \hat{x} denote the unique preimage of x . Let q, r be the two preimages of $f(p)$ other than p . If $f(p) \notin \{a, b, c\}$, then $\{\hat{a}, \hat{b}, \hat{c}\} = [\hat{a}, \hat{c}]$. If $f(p) = c$, then $\{\hat{a}, \hat{b}, q\} = [\hat{a}, q]$. If $f(p) = a$, then \hat{c} covers some $s \in \{p, q, r\}$, and we have $\{s, \hat{b}, \hat{c}\} = [s, \hat{c}]$. If $f(p) = b$, then \hat{c} covers either p or some $s \in \{q, r\}$; accordingly, we have either $\{q, p, \hat{c}\} = [q, \hat{c}]$, or $\{\hat{a}, s, \hat{c}\} = [\hat{a}, \hat{c}]$. In all the cases P contains an interval of cardinality three, which is a contradiction. \square

(b). By (a) it suffices to consider the case where P elementarily zips onto Q along a $p \in P$.

Since the three-point poset $J := f^{-1}(f(p))$ is full in P and J^* is constructible, by Theorem 4.13 it suffices to show that f^* is a filtration map.

If $y \neq f(p)$, then $f^{-1}(y)$ is a singleton. Then $(f^*)^{-1}(\partial \lceil y^* \rceil)$ the boundary of the cone $(f^*)^{-1}(\lceil y^* \rceil)$ and so is of codimension one in it.

It remains to show that $\lceil J^* \rceil \setminus J^*$ is of codimension one in $\lceil J^* \rceil$. Write $J = \{p, q, r\}$. Let m be a maximal element of $\lceil J^* \rceil \setminus J^*$. Suppose that $m < p^*$. Since P is nonsingular, there exists an $n \neq p^*$ such that $m < n < q^*$. This contradicts the maximality of m . Hence $m \not< p^*$. Since $m < q^*$ or $m < r^*$, it follows that m is covered by q^* or r^* . \square

Lemma 4.17. *If P^* is a cell complex and P zips onto Q , then Q^* is a cell complex, and the quotient map $f^*: P^* \rightarrow Q^*$ is a subdivision map.*

For a direct construction of the homeomorphism between $|P|$ and $|Q|$ in the case where P is a manifold see [54; 1.4] or, in the case where P is a sphere, [56; 4.7].

Proof. Arguing by induction, it suffices to consider the case of an elementary zipping along p . If $J = f^{-1}(f(p))$, we claim that $\lceil J^* \rceil$ is a ball with boundary $\lceil J^* \rceil \setminus J^*$.

Indeed, let $J = \{p, q, r\}$. Since P^* is a cell complex, $[x^*]$ is a ball with boundary $\partial[x^*]$ for each $x \in P$. Since J is full in P , $[q^*] \cap [r^*] = [p^*]$. Obviously $[p^*]$ is of codimension one in $[q^*]$ and in $[r^*]$. Hence by basic PL topology [67; Corollary to Theorem 2], $[q^*] \cup [r^*] = [J^*]$ is a ball with boundary $(\partial[q^*] \setminus \{p\}) \cup (\partial[r^*] \setminus \{p\}) = [J^*] \setminus J^*$.

Since P^* is a cell complex, it follows that $f^{-1}([y^*])$ is a ball with boundary $f^{-1}(\partial[y^*])$ for each $y \in Q$. This implies the second assertion of the lemma. Then the first one follows from Theorem 3.4. \square

Lemma 4.18. *If a cell complex P zips onto Q , then Q is a cell complex.*

Proof. We may assume without loss of generality that P zips elementarily onto Q along a $p \in P$. Let $f: P \rightarrow Q$ be the quotient map. Then $\partial[p] \simeq (\partial[f(p)] + S^0)$. Since $\partial[p]$ is a sphere by the hypothesis, and $|\partial[f(p)] + S^0| \cong |\partial[f(p)]| * |S^0|$, by Morton's theorem [52] $\partial[f(p)]$ is a sphere.

Given any $y \in Q$ other than $f(p)$, its preimage $f^{-1}(y)$ is a singleton $\{x\}$. The restriction $f_0: \partial[x] \rightarrow \partial[y]$ of f is either an isomorphism (if $p \not\prec x$) or an elementary zipping along p (if $p \prec x$). Now $\partial[x]$ is a sphere, hence its dual is a cell complex by Theorem 3.46. Then by Lemma 4.17 f_0 is a subdivision. Hence by Theorem 3.4 $\partial[y]$ is a sphere. \square

Theorem 4.19. *If P is a poset such that $P^\#$ is a cell complex, and P zips onto Q , then*

- (a) $Q^\#$ is a cell complex, and
- (b) P transversely zips onto Q .

We recall that $P^\#$ is a cell complex if and only if open intervals $\partial\partial^*[p, q]$ in P are spheres (Lemma 3.42).

Of course, $P^\#$ is a cell complex if either P or P^* is a cell complex (by Corollary 3.48 or alternatively by Lemma 3.42 and Theorem 3.46).

(a). It suffices to show that every open interval in Q is a sphere, or equivalently that $\partial^*[y]$ is a cell complex for every $y \in Q$. Let $X = f^{-1}(\partial^*[y])$.

If $y \neq f(p)$, then $f^{-1}(y)$ is a singleton $\{x\}$, and $X = \partial^*[x]$, which is a cell complex. Also, f restricts either to an isomorphism $X \simeq \partial^*[y]$ or to an elementary zipping $X \rightarrow \partial^*[y]$ along p . In the latter case $\partial^*[y]$ is a cell complex by Lemma 4.18.

Suppose that $y = f(p)$, and let $x \in [p, q, r]$. If $x \not\prec p$, then since $\{p, q, r\}$ is full, either $x \not\prec q$ or $x \not\prec r$. By symmetry we may assume the latter. Then f restricts to an isomorphism between $[q, x]$ and $[y, f(x)]$. Since $\partial\partial^*[q, x]$ is a sphere, so is $\partial\partial^*[y, f(x)]$.

It remains to consider the case $x \succ p$. Then f restricts to an elementary zipping $\partial[x] \rightarrow \partial[f(x)]$ along p . Since $\partial[x]$ is dual to a cell complex, by Lemma 4.17 so is $\partial[f(x)]$. In particular, $\partial\partial^*[y, f(x)]$ is a sphere. \square

(b). By (a) it suffices to consider the case of an elementary zipping $f: P \rightarrow Q$ along a $p \in P$. Let q and r be the two preimages of $f(p)$ other than p , and pick some $x \succ p$. The open interval $\partial\partial^*[q, x]$ is a sphere of some dimension n , hence its dual $\partial\partial^*[x^*, q^*]$ is a cell complex by Theorem 3.46. Its cell $\partial^*[x^*, p^*]$ is maximal, hence is of the same dimension

n . On the other hand, $\partial^*[x^*, q^*]$ is an $(n+1)$ -cell. Thus $|\text{lk}(x^*, [q^*])| \cong C|\text{lk}(x^*, [p^*])|$. Similarly $|\text{lk}(x^*, [r^*])| \cong C|\text{lk}(x^*, [p^*])|$. Thus $[p^*]$ is collared in $[q^*]$ and in $[r^*]$. \square

Lemma 4.20. *If $P^\#$ is a cell complex, and P transversely zips onto Q , then the quotient map $f^*: P^* \rightarrow Q^*$ is transversely constructible.*

Proof. It suffices to consider a transverse elementarily zipping along p . Then it is clear from the definition that $(f^*)^{-1}([y^*])$ is constructible for each $y \in Q$.

Let $M = f^{-1}(\lfloor f(p) \rfloor)$ and $N = f^{-1}(\partial^* \lfloor f(p) \rfloor)$. Let q and r be the two preimages of $f(p)$ other than p .

If $p \in N$ and $x \not\geq p$, then either $\partial[x]_M = \{q\} + \partial[x]_N$ or $\partial[x]_M = \{r\} + \partial[x]_N$. Thus $\text{lk}(x^*, M^*) \simeq C \text{lk}(x^*, N^*)$.

Suppose that $x > p$. Then $\partial[x]$ is dual to a cell complex, and f restricts to an elementary zipping $f_x: \partial[x] \rightarrow \partial[f(x)]$ along p . By Lemma 4.17, f_x^* is a subdivision. Then the pair $(|\text{lk}(x^*, M^*)|, |\text{lk}(x^*, N^*)|)$ is homeomorphic, via the underlying homeomorphism of f_x^* , to the pair $(|\text{lk}(f(x)^*, [f(p)^*])|, |\text{lk}(f(x)^*, \partial[f(p)^*])|)$, which in turn is of the form $(|CX|, |X|)$.

We have thus proved that N^* is collared in N^* . It follows that f^* is a stratification map. \square

From Lemmas 4.20 and 4.16 and Theorem 4.10 we obtain

Theorem 4.21. (a) *If $P^\#$ is a cell complex, and P transversely zips onto a dual cone, then P^* is transversely constructible.*

(b) *If a nonsingular poset P zips onto a dual cone, then P^* is constructible.*

We now turn to a converse type of implication.

Theorem 4.22. (a) *If K is a poset such that K^* is transversely constructible, then K zips onto a dual cone.*

(b) *If K is a cell complex such that K^* is constructible, then K zips onto a singleton.*

(c) *If K is a nonsingular poset such that K^* is transversely constructible, then K transversely zips onto a dual cone.*

It is worth noting that each elementary zipping produced by the proof goes along a σ such that $\partial[\sigma]$ is a prejoin of copies of S^0 (the latter denotes the poset consisting of two incomparable elements).

Proof. We first note that if K is a cell complex and a dual cone, then K is a singleton. Hence by Lemma 4.18, in order to prove (b) it suffices to show that K transversely zips onto a dual cone.

If P is a [transversely] constructible poset, we define its subposet M_P as follows. If P is a singleton, we set $M_P = \emptyset$, and if $P = Q \cup R$ is a [transverse] construction step, we let $M_P = M_Q \cup M_R \cup M_{Q \cap R} \cup M(Q \cap R)$, where $M(P)$ denotes the set of all maximal elements of P . We claim that in the transversely constructible case, $\text{lk}(p, P)$ is a sphere for every $p \in M_P$. To see this, suppose that $P = Q \cup R$ is a transverse construction step,

and for T running over Q , R and $Q \cap R$, and for every $t \in M_T$, $\text{lk}(t, T)$ is a sphere. If $p \in M_{Q \cap R}$ or $p \in M(Q \cap R)$, then $\text{lk}(p, Q \cap R)$ is a sphere (either by the assumption or because it is empty), whence $\text{lk}(p, P) \simeq S^0 * \text{lk}(p, Q \cap R)$ is also a sphere. If $p \in Q \cap R$, then $\text{lk}(p, Q)$ is a cone (over $\text{lk}(p, Q \cap R)$), so it cannot be a sphere; therefore $p \notin M_Q$, and similarly $p \notin M_R$. Hence if $p \in M_Q$, then $\text{lk}(p, P) = \text{lk}(p, Q)$, which is a sphere by the assumption. Similarly, $\text{lk}(p, P)$ is a sphere for each $p \in M_R$, which completes the proof of the claim.

Now in each of (a), (b) we know that K^* is constructible and that $\partial[p]_K$ is a sphere for each $p^* \in M_{K^*}$. Therefore from now on we treat (a) and (b) simultaneously.

For a [transversely] constructible poset P we record its *scheme* of [transverse] construction in the form of a poset S_P , along with a monotone map $s_P: P^* \rightarrow S_P$, defined as follows. If P is a cone, then S_P is a singleton poset, so s_P is uniquely defined. If $P = Q \cup R$ is a [transverse] construction step, we pick some [transverse] construction schemes S_Q , S_R and $S_{Q \cap R}$ and set $S_P = (S_Q \sqcup S_R) + S_{Q \cap R}$ (disjoint union in the concrete category of posets) and $s_P|_{Q \cap R} = s_{Q \cap R}$, $s_P|_{Q \setminus R} = s_Q|_{Q \setminus R}$ and $s_P|_{R \setminus Q} = s_R|_{R \setminus Q}$. It follows by induction that each point-inverse of s_P is a dual cone. (In fact, if P^* happens to be a cell complex, then s_P can be seen to be a bijection – though normally not an isomorphism.) We say that S_T is a (*principal*) *subscheme* of S_P if either $S_T = S_P$ or S_T is a subscheme of S_Q , S_R or $S_{Q \cap R}$ (respectively, of S_Q or S_R), where $P = Q \cup R$ is a [transverse] construction step.

Conversely, let us call a poset S a *scheme poset* if S is a singleton or $S = (S_+ \sqcup S_-) + S_0$, where each S_* is a scheme poset. To each non-atom p of a scheme poset S we associate two subsets $A^+(p, S)$ and $A^-(p, S)$ of S as follows. In the event that $S = (S_+ \sqcup S_-) + S_0$ and p is a non-atom of some S_* (i.e. of S_+ , S_- or S_0), we set $A^+(p, S) = A^+(p, S_*)$ and $A^-(p, S) = A^-(p, S_*)$; and if p is an atom of S_0 , then we define $A^+(p, S)$ to be the set of atoms of S_+ and $A^-(p, S)$ to be the set of atoms of S_- . Given a monotone map $s: P^* \rightarrow S$ whose point inverses are dual cones, let us write $s^{-1}(p) = [p_s]$. Then s corresponds to a construction scheme of P if and only if for each non-atom p of S there exist a $p^+ \in A^+(p, S)$ and a $p^- \in A^-(p, S)$ such that p_s covers both $(p^+)_s$ and $(p^-)_s$. (This is equivalent to the codimension one conditions.)

Now suppose that $X_0 := K^*$ is [transversely] constructible. We may assume that K is not a dual cone; then X_0 is not a cone. If $X_0 = Q \cup R$ is a [transverse] construction step, then either both Q and R are cones, or S_{X_0} contains a proper principal subscheme S_P (namely $P = Q$ or R) such that P is not a cone. By iterating this observation, we can find a principal subscheme S_{P_0} of S_{X_0} such that there is a [transverse] construction step $P_0 = Q_0 \cup R_0$ where both Q_0 and R_0 are cones. Since $X_1 := Q_0 \cap R_0$ is [transversely] constructible, we can similarly find a principal subscheme S_{P_1} of S_{X_1} such that there is a [transverse] construction step $P_1 = Q_1 \cup R_1$ where both Q_1 and R_1 are cones. By iterating, we can find an n such that for each $k = 1, \dots, n$ there is a principal subscheme S_{P_k} of S_{X_k} and a [transverse] construction step $P_k = Q_k \cup R_k$, where both Q_k and R_k are cones, $X_{k+1} = Q_k \cap R_k$, and X_{n+1} is a cone.

The union of the singleton subschemes S_{Q_i} and S_{R_i} , $i = 0, \dots, n$; and $S_{X_{n+1}}$ is a closed subposet Σ of S_{K^*} , isomorphic to $S^0 + \dots + S^0 + pt$ ($n+1$ copies of S^0). We may write out the cones $Q_i = [q_i^*]_{X_i}$ and $R_i = [r_i^*]_{X_i}$ for $i = 0, \dots, n$; and $Q_n \cap R_n = [p^*]_{X_{n+1}}$. The monotone map $s_{K^*}: K \rightarrow S_{K^*}$ sends each q_i onto the unique element of S_{Q_i} , each r_i onto the unique element of S_{R_i} , and p onto the unique element of $S_{X_{n+1}}$. In particular, s_{K^*} restricts to a bijection between the subposet Π of K formed by p and the q_i and the r_i , $i = 0, \dots, n$, and the subposet Σ of S_{K^*} . Since each X_{i+1} is of codimension one in both Q_i and R_i , every atom of X_{i+1}^* covers (in P_i^* , and hence in K) the only atom q_i of Q_i^* as well as the only atom r_i of R_i^* . In particular, p covers both q_n and r_n . Moreover, since each S_{P_i} is a principal subscheme of S_{X_i} , and $P_i = Q_i \cup R_i$ is a construction step, both q_i and r_i are atoms of X_i^* . Hence each of q_i, r_i covers both q_{i-1} and r_{i-1} , for each $i = 1, \dots, n$. Therefore the bijection $\Pi \rightarrow \Sigma$ is an isomorphism, and $(\partial\Pi)^b$ contains an n -simplex that is maximal as a simplex of $(\partial[p])^b$.

Now $p \in M_{K^*}$ so $\partial[p]$ is a sphere, which as we have just shown must be of dimension n . Then it coincides with the n -sphere $\partial\Pi \subset \partial[p]$. In other words, Π is a closed subposet of K . Then $[p] \simeq Q + I$, where Q is an $(n-1)$ -sphere. Finally, since $[q_n^*] \cap [r_n^*] = [p^*]$, the subposet $\{p, q_n, r_n\}$ of K is full in K . Thus we obtain an elementary [transverse] zipping $z: K \rightarrow L$ along p .

Since L has fewer elements than K , it remains to show that L satisfies the hypothesis, namely that L^* is [transversely] constructible and $\partial[p]_L$ is a sphere for each $p^* \in M_{L^*}$ [resp. L is nonsingular]. The latter follows by the proof of Lemma 4.18 [resp. from Lemma 4.16(a)].

To see that L^* is transversely constructible in (c), we note that $z: K \rightarrow L$ is a stratification map by the proof of Lemma 4.20. Let $P = Q \cup R$ be a transverse construction step, where S_P is any subscheme of S_{K^*} and $P \neq P_n$. Then S_P is a principal subscheme of S_{X_i} for some $i < n+1$. If $P = P_i$, then $X := Q \cap R$ contains $\{p, q_n, r_n\}$. If $P \neq P_i$, we claim that X is disjoint from $\{p, q_n, r_n\}$. Indeed, either Q or R contains P_i ; by symmetry we may assume that $P_i \subset Q$. By the above $\text{lk}(p, P_i) \simeq \text{lk}(p, X_i) \simeq S^0 + \dots + S^0$ ($n+1-i$ copies). Hence $\text{lk}(p, Q)$ is a sphere, and in particular not a cone. Therefore $p \notin X$; by a similar argument, $q_n \notin X$ and $r_n \notin X$. Thus $z^{-1}(z(X)) = X$, and it follows from Theorem 3.27(b) that $z(P) = z(Q) \cup z(R)$ is a transverse construction step.

To see that L^* is constructible in (a) and (b), let $S_{L^*} = S_{K^*}/s_{K^*}(I)$, let $z: S_{K^*} \rightarrow S_{L^*}$ be the quotient map, and let $s_{L^*}: L \rightarrow S_{L^*}$ be induced by s_{K^*} . If $z(p)$ is not an atom (which is the case precisely when $n > 0$), then $A^\pm(s_{L^*}z(p), S_{L^*})$ equals the one-to-one image of $A^\pm(s_{K^*}(q_n), S_{K^*})$ or equivalently of $A^\pm(s_{K^*}(r_n), S_{K^*})$, and we may choose $s_{L^*}z(p)^\pm$ to be the images of q_0 and r_0 . If $s_{L^*}z(p) \in A^\pm(s_{L^*}z(x), S_{L^*})$ for some $x \in K$, then $A^\pm(s_{K^*}(x), S_{K^*})$ contains $s_{K^*}(q_n)$ and $s_{K^*}(r_n)$, and does not contain $s_{K^*}(p)$; if x covers q_n or r_n in K , then $z(x)$ covers $z(p)$ in L . It follows that L^* is constructible, with scheme S_{L^*} . \square

Corollary 4.23. *If K is a cell complex and K^* is constructible, then K^* is transversely constructible.*

4.C. Collapsing

4.24. Collapsing of polyhedra. We recall that a polyhedron P is said to *elementarily collapse* onto a subpolyhedron Q if $P = Q \cup R$, where $(R, R \cap Q) \cong (|\Delta^n|, |\Delta^{n-1}|)$ for some n . A *collapse* is a sequence of elementary collapses; P is *collapsible* if it collapses onto a point.

Lemma 4.25. *If P is a transversely constructible poset, then $|P|$ is collapsible.*

Proof. If P is a single cone, then $|P|$ is a cone and hence collapsible. Suppose that $P = Q \cup R$ is a transverse construction step, and let $S = Q \cap R$. Arguing by induction, we may assume that $|Q|$, $|R|$ and $|S|$ are collapsible, say, onto points $\{q\}$, $\{r\}$ and $\{s\}$. Let J_{qs} and J_{rs} be the traces of $\{s\}$ under the collapses $|Q| \searrow \{q\}$ and $|R| \searrow \{r\}$. This yields collapses $|Q| \searrow J_{qs}$ and $|R| \searrow J_{rs}$. Now J_{qs} and J_{rs} are arcs, so they collapse onto $\{s\}$. Thus each of $|Q|$, $|R|$ and $|S|$ collapses onto $\{s\}$.

Since S is collared in Q and in R , there are homeomorphisms

$$|Q| \cong |Q| \bigcup_{|S|=|S| \times \{-1\}} |S| \times [-1, 0] \text{ and } |R| \cong |R| \bigcup_{|S|=|S| \times \{1\}} |S| \times [0, 1],$$

both sending $|S|$ onto $|S| \times \{0\}$ [21; 4.2]. These combine into a homeomorphism

$$|P| \cong |Q| \bigcup_{|S|=|S| \times \{-1\}} |S| \times [-1, 1] \bigcup_{|S| \times \{1\} = |S|} |R|.$$

The given collapse $|S| \searrow \{s\}$ yields a collapse $|S| \times [-1, 1] \searrow \{s\} \times [-1, 1] \cup |S| \times \{-1, 1\}$, so by excision, the right hand side collapses onto

$$|Q| \bigcup_{\{s\}=\{(s,-1)\}} \{s\} \times [-1, 1] \bigcup_{\{(s,1)\}=\{s\}} |R|.$$

Since $|Q|$ and $|R|$ each collapse onto $\{s\}$, the latter collapses onto $\{s\} \times [-1, 1]$ — which is collapsible. \square

Proposition 4.26. *If P is a constructible 2-dimensional simplicial complex, then $|P| \times [0, 1]$ is collapsible.*

Proof. Let $P = Q \cup R$ be a construction step. Arguing by induction, we may assume that $|Q| \times [0, 1]$ and $|R| \times [0, 1]$ are collapsible. Clearly, $|P| \times I$ collapses onto $X := |Q| \times [0, \frac{1}{3}] \cup |Q \cap R| \times [\frac{1}{3}, \frac{2}{3}] \cup |R| \times [\frac{2}{3}, 1]$. Since $Q \cap R$ is a transversely constructible 1-dimensional simplicial complex, $|Q \cap R|$ is a tree, so it is collapsible. Hence X collapses onto $Y := |Q| \times [0, \frac{1}{3}] \cup pt \times [\frac{1}{3}, \frac{2}{3}] \cup |R| \times [\frac{2}{3}, 1]$. Let T_1 be the trace of $(pt, \frac{1}{3})$ under the collapse of $|Q| \times [0, \frac{1}{3}]$ onto a point, and let T_2 be the trace of $(pt, \frac{2}{3})$ under the collapse of $|R| \times [\frac{2}{3}, 1]$ onto a point. Then Y collapses onto the tree $T_1 \cup pt \times [\frac{1}{3}, \frac{2}{3}] \cup T_2$. \square

The preceding proposition implies that if P is a constructible 2-dimensional simplicial complex, then $|P|$ 3-deforms to a point; it follows that $|P|$ embeds in the 4-ball (see [1]).

See [11] concerning implications between variants of constructibility and collapsibility in the case of triangulated balls.

4.27. Simplicial collapsing. We recall that a simplicial complex K is said to *elementarily simplicially collapse* onto a proper subcomplex L if $K = L \cup [\sigma]$ for some $\sigma \in K$

such that $Q \cap \lceil \sigma \rceil = \{v\} * \partial \lceil \tau \rceil$, where $\sigma = \{v\} * \tau$ for some vertex $v \in \sigma$. The simplicial complex K is said to *simplicially collapse* onto a subcomplex L if either $K = L$ or K elementarily simplicially collapses onto a subcomplex that simplicially collapses onto L . Finally, K is defined to be *simplicially collapsible* if it simplicially collapses onto a singleton.

It is well-known (see [66], [67], [57]) that a polyhedron P collapses onto a subpolyhedron Q if and only if (P, Q) admits a triangulation by a pair (K, L) of simplicial complexes such that K simplicially collapses onto L .

4.28. Collapsing of posets. Let us give an inductive definition of a collapsible poset. A poset P is said to *elementarily collapse* onto a proper closed subposet Q if $P = Q \cup \lceil \sigma \rceil$ for some $\sigma \in P$ and $Q \cap \lceil \sigma \rceil$ is collapsible. P is said to *collapse* onto a closed subposet Q if either $P = Q$ or P elementarily collapses onto a closed subposet R that collapses onto Q . Finally, P is defined to be *collapsible* if it collapses onto a singleton poset.

Note that a cone collapses onto its maximal element. We have the following obvious excision rule for collapsing: if P is a poset and Q is a closed subposet of P , then P collapses onto Q if and only if $P \setminus R$ collapses onto $Q \setminus R$, where $R = P \setminus \lceil P \setminus Q \rceil$.

By definition, simplicial collapsibility implies collapsibility; the converse implication holds (not only for simplicial complexes) after passing to the barycentric subdivision:

Lemma 4.29. *If a poset P collapses onto a closed subposet Q , then P^b simplicially collapses onto Q^b .*

We use the notation $\hat{\sigma}$ for the barycenter of a $\sigma \in P$, that is, the vertex of P^b formed by the singleton chain $\{\sigma\}$.

Proof. Without loss of generality the given collapse is elementary, with $P = Q \cup \lceil \sigma \rceil$ and $Q \cap \lceil \sigma \rceil$ collapsible onto a singleton poset $\{p\}$. Arguing by induction, we may assume that $(Q \cap \lceil \sigma \rceil)^b$ collapses simplicially onto the vertex $\{\hat{p}\}$.

Now $\lceil \sigma \rceil^b = \{\hat{\sigma}\} * (\partial \lceil \sigma \rceil)^b$ collapses simplicially onto the subjoin $\{\hat{\sigma}\} * (Q \cap \lceil \sigma \rceil)^b$, which in turn collapses simplicially onto $\{\hat{\sigma}\} * \{\hat{p}\} \cup (Q \cap \lceil \sigma \rceil)^b$ using the previously constructed simplicial collapse. The latter collapses simplicially onto $(Q \cap \lceil \sigma \rceil)^b$, so by excision, P^b simplicially collapses onto Q^b . \square

4.30. Shelling. A poset P is inductively defined to be *elementarily [transversely] shellable* onto a proper closed subposet Q if $P = Q \cup CX$, where CX is a cone of P , such that $R := Q \cap CX$ is [transversely] shellable and is of codimension one [resp. collared] in Q , and $R \cap \lceil X \setminus R \rceil$ is of codimension one [resp. collared] in R and in $\lceil X \setminus R \rceil$. (Clearly [by Lemma 3.23(a)], this implies that R is also of codimension one [collared] in CX .) P is said to *[transversely] shell* onto a closed subposet Q if either $P = Q$ or P elementarily [transversely] shells onto a closed subposet R that [transversely] shells onto Q . Finally, P is defined to be *[transversely] shellable* if it [transversely] shells onto a cone.

[Transversely] shellable posets are both collapsible (onto a cone, and hence onto a point) and [transversely] constructible. In particular, \emptyset is not shellable.

The following is proved similarly to Lemma 4.4:

Lemma 4.31. *Let P be a poset with pure cones. Then C^*P is shellable if and only if either P is a cone or the empty set, or $P = Q \cup CX$, where Q is a proper closed subposet of P and CX is a cone of P , such that Q and $R := Q \cap CX$ are shellable, and if CX is n -dimensional, then Q is also n -dimensional, R is $(n - 1)$ -dimensional, $\lceil X \setminus R \rceil$ is purely $(n - 1)$ -dimensional, and $R \cap \lceil X \setminus R \rceil$ is purely $(n - 2)$ -dimensional.*

It follows that if K is an affine polytopal complex, then the shellability of C^*K implies “shellability” of K in the sense of Bruggesser and Mani [17] and is implied by “shellability” of K in the sense of Björner and Wachs (see [70], [37]). Note that C^*K is precisely the “face poset” of K in sense of Topological Combinatorics (where the empty set is regarded a face).

Example 4.32. If P is an n -dimensional convex polytope in \mathbb{R}^n , let us consider its boundary dP and the part dP_v of dP that is visible from some $v \in \mathbb{R}^m \setminus P$. Then the face poset of dP (including the empty face), and the poset of nonempty faces of dP_v for every $v \in \mathbb{R}^m \setminus P$ are shellable (see [70; Theorem 8.12]).

Remark 4.33. If K is an n -dimensional simplicial complex, then there are two well-known characterizations of shellability of its face poset C^*K (in this case the Bruggesser–Mani and Björner–Wachs definitions are equivalent; see [70], [15]):

- (i) $K = K_0$ can be reduced to a single n -simplex by elementary reductions of the form $K_{i+1} = K_i \cup \lceil \sigma_i \rceil$, where $\dim |K_i \cap \lceil \sigma_i \rceil| = n - 1$ for each i , and $(\lceil \sigma_i \rceil, K_i \cap \lceil \sigma_i \rceil) \simeq (\Delta^k * \Delta^{n-k-1}, \partial \Delta^k * \Delta^{n-k-1})$ for some $k = k(i)$ such that $0 \leq k \leq n$.
- (ii) $K = K_0$ is purely n -dimensional and can be reduced to a single n -simplex by elementary reductions of the form $K_{i+1} = K_i \cup \lceil \sigma_i \rceil$, where each $\dim \lceil \sigma_i \rceil = n$ and each $K_i \cap \lceil \sigma \rceil$ is purely $(n - 1)$ -dimensional (and nonempty).

We note that the shellability of K itself can be characterized as in (i), but with $k \neq n$ (see also the proof of Lemma 4.5(c)); this definition of shellability for simplicial complexes is found in [46; 5.2].

Similarly to Lemma 4.6, one has (using [67; Theorem 6]):

Lemma 4.34. (a) *If a pseudo-manifold N shells onto a manifold M , then $|N|$ is homeomorphic to $|M|$.*

(b) *If a cell complex N transversely shells onto a manifold M , then $|N|$ is homeomorphic to $|M|$.*

Similarly to Lemma 4.8, one has

Lemma 4.35. *If P is a [transversely] shellable poset, then $\lfloor p \rfloor$ is [transversely] shellable for each $p \in P$.*

The proof of Lemma 4.12 also works to establish the following lemma (whose $\lceil L^* \rceil_{K^*}$ corresponds to K^* of Lemma 4.12).

Lemma 4.36. *Let K be a poset, L a full subposet of K , and M a closed subposet of L .*

(a) *If L is closed in K , and $\lceil L^* \rceil_{K^*}$ [transversely] shells onto $\lceil M^* \rceil_K$, then L^* [transversely] shells onto M^* .*

(b) *If L^* is shells onto M^* , then $\lceil L^* \rceil_{K^*}$ shells onto $\lceil M^* \rceil_{K^*}$.*

4.37. Excision. We have the following obvious excision rule for shelling: if P is a poset and Q is a closed subposet of P , then P [transversely] shells onto Q if and only if $P \setminus R$ [transversely] shells onto $Q \setminus R$, where $R = P \setminus \lceil P \setminus Q \rceil$. Since $P \setminus R$ is open in P , by Lemma 3.31 the inclusion $P \setminus R \subset P$ is a stratification map. Theorem 3.27 [resp. 3.37] implies the following stronger “map excision” rule:

Lemma 4.38. *Suppose that $f: P_1 \rightarrow P_2$ is a stratification [resp. filtration] map, Q_i is a closed subposet respectively of P_i for $i = 1, 2$ such that $f^{-1}(Q_2) = Q_1$ and f restricts to an isomorphism between $P_1 \setminus \text{Int } Q_1$ and $P_2 \setminus \text{Int } Q_2$. Then P_1 [transversely] shells onto Q_1 if and only if P_2 [transversely] shells onto Q_2 .*

Lemma 4.39. *Let P and Z be posets, Q a closed subposet of P such that $Q \cap \lceil P \setminus Q \rceil$ is collared in Q and in $\lceil P \setminus Q \rceil$, and R a closed subposet of Q . If $Q \times CZ \cup CQ \times Z$ transversely shells onto $R \times CZ \cup CQ \times Z$, then $Q \times CZ \cup CP \times Z$ transversely shells onto $R \times CZ \cup CP \times Z$.*

Proof. Define a retraction $\varphi: CP \rightarrow CQ$ by sending $P \setminus Q$ onto the maximal element of CQ . By Lemma 3.23(a), φ is a stratification map. Define a retraction $f: Q \times CZ \cup CP \times Z \rightarrow Q \times CZ \cup CQ \times Z$ by $f = \text{id}_{Q \times CZ} \cup \varphi \times \text{id}_Z$. Then f is a stratification map, and the assertion follows from Lemma 4.38. \square

4.40. \emptyset -shelling. A poset P is defined to be [transversely] \emptyset -shellable if it \emptyset -shells onto the empty set, where an (elementary) [transverse] \emptyset -shelling is defined similarly to an (elementary) [transverse] shelling.

We note that every cone elementarily transversely \emptyset -shells onto \emptyset , so [transverse] \emptyset -shellability is strictly weaker than [transverse] shellability.

It is easy to see that the product of transversely shellable posets is transversely shellable. The following lemma is concerned with the cojoin, $P \bar{*} Q \simeq P \times CQ \cup CP \times Q$.

Lemma 4.41. *Suppose that a poset P transversely \emptyset -shells onto a closed subposet R , and let Z be a transversely shellable poset. Then $P \times CZ \cup CP \times Z$ transversely shells onto $R \times CZ \cup CP \times Z$.*

In particular, the cojoin of a transversely \emptyset -shellable poset and a transversely shellable poset is transversely shellable.

We will only need the case $Z = pt$ of this lemma.

Proof. Suppose that P elementarily transversely \emptyset -shells onto Q , where Q transversely \emptyset -shells onto R . Arguing by induction, we may assume that $Q \times CZ \cup CQ \times Z$ transversely shells onto $R \times CZ \cup CQ \times Z$. Then by Lemma 4.39, $Q \times CZ \cup CP \times Z$ transversely shells onto $R \times CZ \cup CP \times Z$. Thus it suffices to show that $P \times CZ \cup CP \times Z$ transversely shells onto $Q \times CZ \cup CP \times Z$.

We have $P = Q \cup CY$, where $S := Q \cap CY = Q \cap Y$ is transversely \emptyset -shellable and collared in Q , and $S \cap \lceil Y \setminus S \rceil$ is collared in S and in $\lceil Y \setminus S \rceil$ (and so by Lemma 3.23(a), S is also collared in CY). Then $CY \times CZ$ is a single cone, which meets $Q \times CZ \cup CP \times Z$ in $S \times CZ \cup CY \times Z$. By the second assertion of Lemma 3.23(b), $S \times CZ \cup CY \times Z$ is collared in $Q \times CZ \cup CP \times Z$. Next, we have $\lceil (Y \times CZ \cup CY \times Z) \setminus (S \times CZ \cup CY \times Z) \rceil = \lceil Y \setminus S \rceil \times CZ$. By the above $(\lceil Y \setminus S \rceil \cap S) \times CZ$ is collared in $\lceil Y \setminus S \rceil \times CZ$ and in $S \times CZ$. Then by the first assertion of Lemma 3.23(b) it is also collared in $S \times CZ \cup CY \times Z$.

It remains to show that $S \times CZ \cup CY \times Z$ is shellable. Since S is transversely \emptyset -shellable, we may assume by induction that $S \times CZ \cup CS \times Z$ transversely shells onto $CS \times Z$. Since $S \cap \lceil Y \setminus S \rceil$ is collared in S and in $\lceil Y \setminus S \rceil$, by Lemma 4.39, $S \times CZ \cup CY \times Z$ transversely shells onto $CY \times Z$. Since Z is transversely shellable, so is $CY \times Z$. \square

Lemma 4.42. *Let X be a poset, Y a closed subposet of X , and Z a closed subposet of Y . Then $\lceil (Y^b)^* \rceil_{(X^b)^*}$ transversely \emptyset -shells onto $\lceil (Z^b)^* \rceil_{(X^b)^*}$.*

Proof. By arranging the elements of $Y \setminus Z$ in a total order extending the original partial order, it suffices to consider the case where $Y = Z \cup \lceil \sigma \rceil_X$, with $\partial \lceil \sigma \rceil_X \subset Z$. Then $H_\sigma = CA \times CB$, where $A \simeq H(\partial \lceil \sigma \rceil)$ and $B \simeq H(\partial \lceil \sigma^* \rceil)$. We have $H_\sigma \cup \lceil (Z^b)^* \rceil_{(X^b)^*} = \lceil (Y^b)^* \rceil_{(X^b)^*}$ and $H_\sigma \cap \lceil (Z^b)^* \rceil_{(X^b)^*} = A \times CB$. Arguing by induction, we may assume that A is transversely \emptyset -shellable, and hence so is $A \times CB$. \square

Theorem 4.43. *Let M be a poset and P a closed subposet of M . If P collapses onto a closed subposet Q , then $N(P, M) := \lceil H(P) \rceil_{H(M)}$ shells onto $N(Q, M)$.*

In particular, if P is collapsible, then $N(P, M)$ is shellable (and hence constructible).

Proof. It suffices to consider the case of an elementary collapse. Thus suppose that $P = Q \cup \lceil \sigma \rceil$, where $R := Q \cap \lceil \sigma \rceil$ is collapsible, $R \subset \partial \lceil \sigma \rceil$. Arguing by induction, we may assume that $N(R, \partial \lceil \sigma \rceil)$ is shellable.

Since σ is maximal in P , ∂H_σ^P is isomorphic to $H(\partial \lceil \sigma \rceil)$. This isomorphism takes $N(Q, P) \cap H_\sigma^P$ onto $N(R, \partial \lceil \sigma \rceil)$. Since the latter is shellable, it follows that $N(Q, P) \cup H_\sigma^P$ shells onto $N(Q, P)$.

On the other hand, $H(\lceil \sigma \rceil)$ is isomorphic to the cojoin $H(\partial \lceil \sigma \rceil) \times C(pt) \cup CH(\partial \lceil \sigma \rceil) \times pt$. By Lemma 4.42, $H(\partial \lceil \sigma \rceil)$ transversely \emptyset -shells onto $N(R, \partial \lceil \sigma \rceil)$. Hence by Lemma 4.41, the cojoin $H(\partial \lceil \sigma \rceil) \times C(pt) \cup CH(\partial \lceil \sigma \rceil) \times pt$ transversely shells onto $N(R, \partial \lceil \sigma \rceil) \times C(pt) \cup CH(\partial \lceil \sigma \rceil) \times pt$. The isomorphism sends the latter onto $N(R, \lceil \sigma \rceil) \cup H_\sigma^P$. Thus $H(\lceil \sigma \rceil)$ transversely shells onto $N(R, \lceil \sigma \rceil) \cup H_\sigma^P$. By excision (using that $\text{Fr}(N(Q, P) \cup H_\sigma^P)$ is disjoint from Q), we get that $N(P, P)$ transversely shells onto $N(Q, P) \cup H_\sigma^P$.

Thus $N(P, P)$ shells onto $N(Q, P)$. Hence by Lemma 4.36, $N(P, M)$ shells onto $N(Q, M)$. \square

Corollary 4.44. *If a poset P is collapsible, then $H(P) = (P^\flat)^*$ is shellable.*

Corollary 4.45. *If P collapses onto a closed subposet Q such that $H(Q)$ is constructible, then $H(P)$ is constructible.*

Proof. Since $N(Q, Q) = H(Q)$ is constructible, by Lemma 4.12, so is $N(Q, P)$. Hence $H(P) = N(P, P)$ is constructible by Theorem 4.43. \square

4.D. Collapsible maps

Lemma 4.46. *If X is a simplicial complex such that $|X|$ is collapsible, then X admits a simplicially realizable subdivision map $\beta: X' \rightarrow X$ such that X' is simplicially collapsible and $\beta^{-1}(Y)$ is simplicially collapsible for each simplicially collapsible subcomplex Y of X .*

Proof. We use Whitehead's theorem [66; Theorem 7] (see also [33; Theorem III.6]) implying that X admits a stellar subdivision map $\beta: X' \rightarrow X$ such that X' is a simplicially collapsible. It is easy to see by considering elementary stellar subdivisions that a stellar subdivision of a simplicially collapsible simplicial complex is simplicially collapsible [66; Theorem 4], and the assertion follows. \square

Remark 4.47. An alternative proof could be based on a recent result by Adiprasito and Benedetti [3; Corollary 3.5] implying that if $\beta: X' \rightarrow X$ is a simplicially realizable subdivision map, then there exists an m such that $\gamma_m^{-1}(Y)$ is simplicially collapsible for each simplicially collapsible subcomplex Y of X , where $\gamma_m: (X')^{bm} \xrightarrow{bm} X' \xrightarrow{\beta} X$ is the m th barycentric subdivision map.

Theorem 4.48. *Let $f: K \rightarrow L$ be a simplicial map such that $|f|$ is collapsible. Then there exists a simplicially realizable subdivision map $\alpha: K' \rightarrow K^\flat$ such that the composition $g: K' \xrightarrow{\alpha} K^\flat \xrightarrow{f^\flat} L^\flat$ is simplicial and dual to a transversely constructible map.*

That g be simplicial is equivalent to saying that $\alpha((K')^{(0)}) \subset (f^\flat)^{-1}((L^\flat)^{(0)})$.

Proof. Let $\sigma_1, \dots, \sigma_r$ be the simplices of L arranged in an order of increasing dimension. Let $F_i = f^{-1}(\sigma_i)$. The simplicially realizable subdivision map $\beta_i: F_i^\beta \rightarrow F_i^\flat$ given by Lemma 4.46 followed by the barycentric subdivision map $\flat: (F_i^\beta)^\flat \rightarrow F_i^\beta$ extends by taking joins with the links to a simplicially realizable subdivision map $K_i \rightarrow K_{i-1}$, where $K_0 = K^\flat$. Let $\alpha: K' \rightarrow K^\flat$ be the resulting subdivision map, where $K' = K_r$. It remains to show that g^* is transversely constructible.

It is easy to see that the barycentric subdivision of a simplex is collapsible. Hence by Lemma 4.46, $\lceil \rho \rceil^\beta := \beta_i^{-1}(\lceil \rho \rceil^\flat)$ is collapsible for each $\rho \in F_i$ and each i .

Let $K = K_0 \xrightarrow{f_1} \dots \xrightarrow{f_r} K_r = L$ be the factorization of f given by Lemma 2.74, where f_i shrinks F_i onto an element $p_i \in K_i$. Let $\hat{p}_i = (p_i) \in K_i^\flat$. It is not hard to see that the fibers $\Phi_\tau = (f_i^\flat)^{-1}(\tau)$, $\tau \in K_i^\flat$ of the simplicial map $f_i^\flat: K_{i-1}^\flat \rightarrow K_i^\flat$ are $\Phi_{\hat{p}_i} = F_i^\flat$ and $\Phi_\tau \simeq \lceil \rho_i \rceil^\flat$, where $\tau = (\dots > \rho_j > p_i)$, necessarily with $\rho_j \in F_j$, where σ_j covers σ_i in K , where ρ_i is the image of ρ_j under the Hatcher map $f_{\sigma_j \sigma_i}: F_j \rightarrow F_i$.

Let $K' = K'_0 \xrightarrow{g_1} \dots \xrightarrow{g_r} K'_r = L^\flat$ be the obvious simplicial factorization of g given by the simplicial factorization of f^\flat . Thus g_i shrinks $(F_i^\beta)^\flat$ onto an element $\hat{p}'_i \in K'_i$. Clearly, the fibers $\Phi'_\tau = g_i^{-1}(\tau)$, $\tau \in K'_i$ of the simplicial map g_i are $\Phi'_{\hat{p}'_i} = (F_i^\beta)^\flat$ and $\Phi'_\tau \simeq ([\rho_i]^\beta)^\flat$, where $\tau = (\dots > \rho'_j > p_i)$, necessarily with $\rho'_j \in [\rho_j]^\beta$, $\rho_j \in F_j$, where σ_j covers σ_i in K , where ρ_i is the image of ρ_j under the Hatcher map $f_{\sigma_j \sigma_i}: F_j \rightarrow F_i$.

Since F_i^β and $[\rho_i]^\beta$ are collapsible, by Corollaries 4.45 and 4.23, $H(F_i^\beta)$ and $H([\rho_i]^\beta)$ are transversely constructible. Since g_i is simplicial, by Corollary 3.54, g_i^* is a stratification map. Hence g_i^* is transversely constructible. Thus by Theorem 4.11 g^* is transversely constructible. \square

Theorem 4.49. *Let P be a polyhedron and $f: P \rightarrow Q$ a collapsible map onto a collapsible polyhedron. Then P is collapsible.*

Proof. Let $\varphi: K \rightarrow L$ be a simplicial map triangulating f . Let L' be a collapsible (e.g. simplicially collapsible) affine simplicial subdivision of L . By Corollaries 4.45 and 4.23, $H(L')$ is transversely constructible.

On the other hand, by basic PL topology there is a simplicial affine subdivision K' of K and a simplicial map $f: K' \rightarrow L'$ triangulating f . By Theorem 4.48, there exists a simplicially realizable subdivision map $\alpha: K'' \rightarrow (K')^\flat$ such that the composition $K'' \xrightarrow{\alpha} (K')^\flat \xrightarrow{f^\flat} (L')^\flat$ is simplicial, and its dual is transversely constructible. Then by Lemma 4.10, $(K'')^*$ is transversely constructible. Hence by Lemma 4.25 P is collapsible. \square

Remark 4.50. Let us also sketch a much easier (but not necessarily combinatorially clearer) proof of Theorem 4.49. Let K be a simplicially collapsible subdivision of a triangulation of Q that makes f simplicial. Then f is triangulated by a simplicial map $L \rightarrow K$. Given an elementary simplicial collapse of K onto $K \setminus \{\sigma, \tau\}$, where σ is a maximal element of K and τ a maximal element of $K \setminus \{\sigma\}$, we can collapse P onto $f^{-1}(|K \setminus \{\sigma\}|) \cup X$, where X is an appropriate “section” of f over $|\sigma|$, using the Hatcher decomposition (Theorem 2.73). The latter in turn collapses onto $f^{-1}(|K \setminus \{\sigma, \tau\}|) \cup X$, which finally collapses onto $f^{-1}(|K \setminus \{\sigma, \tau\}|)$.

Corollary 4.51. *Composition of collapsible maps is collapsible.*

M. M. Cohen showed that composition of collapsible retractions is a collapsible retraction [21; 8.6].

The following corollary of Theorems 4.14, 3.4 and 4.49 is originally due to T. Homma (see [59; Lemma 5.4.1]) and M. M. Cohen [20].

Corollary 4.52 (Cohen–Homma). *If M is a closed manifold, X is a polyhedron, and there exists a collapsible map $M \rightarrow X$, then X is homeomorphic to M .*

A much easier (but not necessarily combinatorially clearer) proof of Corollary 4.52 is given by Theorem 3.4, the proof of Theorem 4.14 and classical results about collapsing.

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